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Vectors

INTRODUCTION

A particle confined to a straight line (one dimensional motion) can move in only two directions *viz.* either upward or downward from the selected origin or towards right or left from the origin. We can take the motion of the particle to be positive in one of these directions and negative in the other direction. Therefore, in one dimensional motion, the vector quantities like displacements, velocities and accelerations can have either positive or negative values. We can use simple arithmetic to add or subtract vector quantities in this case. However, the situation is not that simple in two or three dimensional motion. For example, in case of two dimensional motion, the particle moves in a plane –say XY plane. The position of the particle is described by two numbers *viz.* x and y coordinates. Suppose a particle is at point P_1 (x_1, y_1) at any instant and after some time it is at another point P_2 (x_2, y_2). The vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ (O is the origin of the coordinate system) are not along the same line as was the case with one dimensional motion. To carry out the addition or subtraction of vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, we require special methods. This is called *vector algebra*. Therefore, in order to study two dimensional motion (*i.e.*, motion in a plane) and three dimensional motion (*i.e.*, motion in space), we need the help of vector algebra. In this chapter, we shall discuss the various properties of vectors.

8.1. VECTOR

The physical quantities in physics are divided into two groups *viz. scalars* and *vectors*. Those physical quantities which have magnitude only are called scalars *e.g.*, mass, length, temperature, speed *etc.* The physical quantities which have magnitude and direction are called vectors *e.g.*, displacement, velocity, acceleration *etc.* If a physical quantity has both magnitude and direction but does not obey the laws of vector addition, it will not be called a vector. For example, electric current in a wire has both magnitude and direction but it does not obey the laws of vector addition. Therefore, electric current is not a vector.

A physical quantity that has both magnitude and direction and also obeys the laws of vector addition is called a vector.

A vector is represented *graphically* by a straight line with an arrowhead as shown in Fig. 8.1. The length of line (OA) represents the magnitude of the vector (on suitable scale) and the arrowhead indicates its direction.

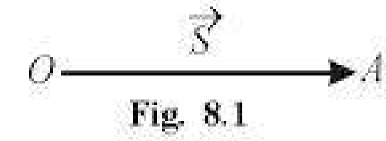


Fig. 8.1

Thus in Fig. 8.1, displacement vector $(\vec{S})$ is represented graphically. The starting point (O) of the arrowed line is called **tail of the vector** and the end (A) of the arrowed line is called **head** or **tip of the vector**.

In *writing*, a vector is represented by a single letter with an arrow on it. Thus in Fig. 8.1, displacement vector will be written as $\vec{S}$. The magnitude of the vector is represented by the same letter used for the vector without arrow. For example, the magnitude of displacement vector $\vec{S}$ is represented as S or $|\vec{S}|$.

Discussion. The following points are worth noting :

(*i*) The magnitude of a vector is a scalar and is always positive.

(*ii*) The magnitude of a vector is also called *modulus* of the vector and is represented by enclosing the vector symbol between two vertical lines. For example, the modulus of displacement vector $\vec{S}$ will be represented as $|\vec{S}|$.

(*iii*) Vectors can be added, subtracted and multiplied. *However, division of a vector by another vector is not a valid operation in vector algebra.* It is because the division of a vector by a direction is not possible.

8.2. SOME DEFINITIONS IN VECTOR ALGEBRA

(*i*) Equal vectors. *The two vectors are said to be equal if they have the same magnitude and the same direction.*

In Fig. 8.2, the two vectors $\vec{A}$ and $\vec{B}$ are equal vectors because they have the same magnitude and the same direction. For two vectors to be equal, it is not necessary that their tails should have the same starting location. Equal vectors may have different locations but they must have the same magnitude and the same direction. Thus in Fig. 8.2(*i*), the two equal vectors $\vec{A}$ and $\vec{B}$ have the same starting location (O). However, in Fig. 8.2(*ii*), the two equal vectors $\vec{A}$ and $\vec{B}$ have different starting locations (O). *This property of equal vectors is important because we can shift a vector from one location to another in a diagram provided its magnitude and direction are not changed.*

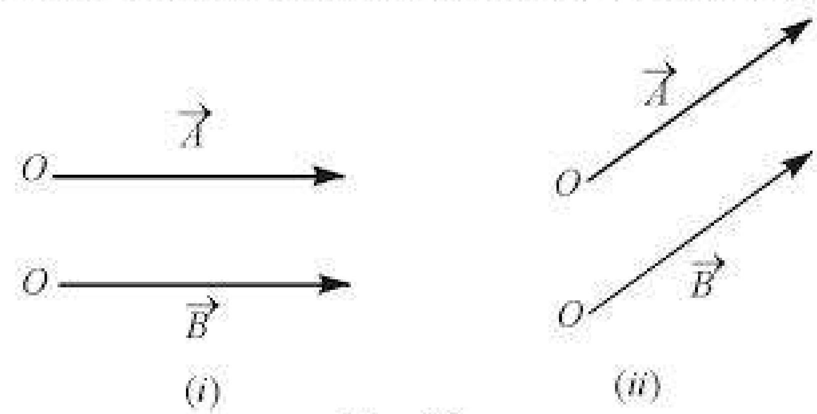


Fig. 8.2

(*ii*) Negative of a vector. *A vector is said to be negative vector of a given vector if its magnitude is the same as that of the given vector but its direction is opposite.*

The negative vector of $\vec{A}$ is represented as $-\vec{A}$ as shown in Fig. 8.3. Note that vectors $\vec{A}$ and $-\vec{A}$ have the same magnitude but opposite directions. In other words, the vectors have the same length but opposite directions. In order to find the negative of a vector, we merely reverse its arrowhead. Clearly, the addition of a vector to its own negative vector will result in zero magnitude *i.e.*,

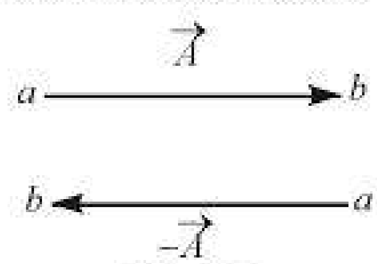


Fig. 8.3

$$\vec{A} + (-\vec{A}) = 0$$

This also applies to ordinary numbers. For example, the negative of 3 is –3 and their sum = 3 + (–3) = 0.

(*iii*) Unit vector. In vector algebra, it is often useful to represent a vector by its magnitude and a *unit vector.* A unit vector is a dimensionless vector that has a magnitude of 1 and has the same direction as that of the given vector.

A unit vector of the given vector is a vector of unit magnitude and has the same direction as that of the given vector.

For example, any vector $\vec{A}$ can be written as :

$$\vec{A} = A\hat{A}$$

Here A is the magnitude of $\vec{A}$ and $\hat{A}$ is the unit vector whose magnitude is 1 and direction is the same as that of $\vec{A}$. Note that *unit vector* $\hat{A}$ *is read as 'A cap' or 'A hat'.*

A unit vector is used to specify a given direction. Unit vectors have no other physical significance. Vectors can be conveniently written in terms of unit vectors which point along the chosen coordinate axes. In rectangular coordinate system, these unit vectors are called $\hat{i}$, $\hat{j}$ and $\hat{k}$; they point respectively along the positive X, Y and Z axes as shown in Fig. 8.4. Since unit vectors have magnitude of 1, they may be used to give direction to vectors. For example, if a force (vector) of 6 N is acting along the + X direction, we can represent the force as $6\hat{i}$ N. Similarly, if an object has a velocity (vector) of $-5\hat{j}$ ms^{-1}, it means that magnitude of velocity is 5 ms^{-1} and its direction is along the –ve Y direction. Note that the only purpose of unit vectors is to indicate the direction of a vector.

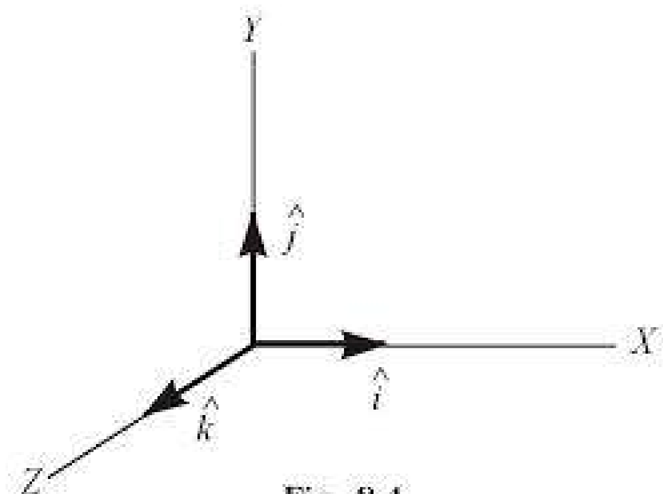


Fig. 8.4

(*iv*) Zero vector. *A vector that has zero magnitude is called zero vector or null vector.*

A zero vector or null vector is represented by $\vec{0}$ (arrow over the number zero). There is no need to specify the direction of a zero vector as its length is zero. The result of addition of a vector to its own negative vector is a zero vector *i.e.,* $\vec{A} + (-\vec{A}) = \vec{0}$. Similarly, when a vector is multiplied by zero, the result is a zero vector *i.e.,* $0(\vec{A}) = \vec{0}$.

Examples. (*a*) The velocity vector of a stationary object is a zero vector.

(*b*) The acceleration vector of an object moving with uniform velocity is a zero vector.

(*c*) The position vector of the origin of the coordinate axes is a zero vector.

8.3. POSITION VECTOR AND DISPLACEMENT VECTOR IN A PLANE

Consider the motion of an object in the XY plane with origin at O. Suppose at any time t_1, the object is at point $P_1(x_1, y_1)$ and at time t_2, it is at point $P_2(x_2, y_2)$.

1. Position vector. *The* ***position vector*** *of an object is the vector from the origin O of the coordinate system to the position of the object.*

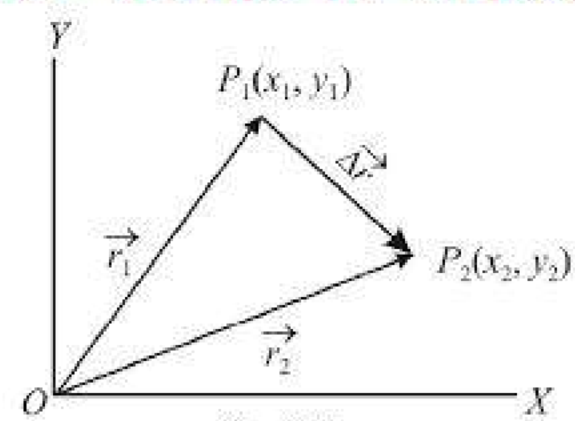


Fig. 8.5

Thus in Fig. 8.5, the position vector of the object at point P_1 is $\overrightarrow{OP_1} (= \vec{r_1})$ while the position vector of the object at point P_2 is $\overrightarrow{OP_2} (= \vec{r_2})$. The position vector of an object provides two important informations :

(*i*) *It gives straight line distance of the object from the origin O (starting point).*

(*ii*) *It gives the direction of the object w.r.t. origin.*

2. Displacement vector. Displacement is a vector quantity and is called displacement vector. We know that the displacement of an object is the straight line distance (shortest distance) between the initial position (P_1) and the final position (P_2) of the object. Thus in Fig. 8.5, $\overrightarrow{P_1P_2} (= \overrightarrow{\Delta r})$ is the displacement vector. The straight line distance P_1P_2 is the magnitude of the displacement vector and the direction of the displacement vector is from P_1 to P_2.

Hence displacement vector is a vector that points from object's initial position to its final position and whose magnitude is equal to the straight line distance between the two points.

$$\text{Position vector, } \overrightarrow{OP_1} = x_1\hat{i} + y_1\hat{j}$$

$$\text{Position vector, } \overrightarrow{OP_2} = x_2\hat{i} + y_2\hat{j}$$

$$\therefore \quad \text{Displacement vector, } \overrightarrow{P_1P_2} = \text{Final position} - \text{Initial position}$$

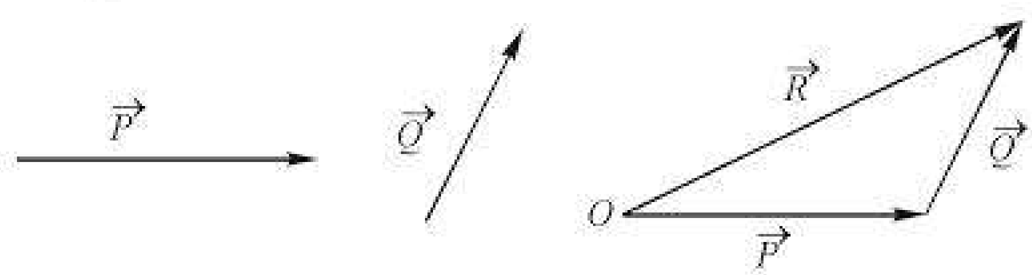
$$= \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= (x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})$$

$$\therefore \quad \overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

8.4. ADDITION OF VECTORS

Vector addition can be done most easily by a graphical method. Suppose we want to add two vectors $\vec{P}$ and $\vec{Q}$ (suppose these are displacement vectors) as shown in Fig. 8.6(*i*). To add them graphically, we place the tail of vector $\vec{Q}$ at the head (or tip) of vector $\vec{P}$ as shown in Fig. 8.6(*ii*). Note that we can move a vector from one place to another but we cannot change the magnitude and direction of the vector while moving it. Having moved vector $\vec{Q}$, we draw another vector $\vec{R}$ from that tail of vector $\vec{P}$ to the head of vector $\vec{Q}$. This vector $\vec{R}$ represents the net displacement (or vector sum of $\vec{P}$ and $\vec{Q}$) and is called the **resultant** of vectors $\vec{P}$ and $\vec{Q}$. We can write the resultant in the form of mathematical equation as :

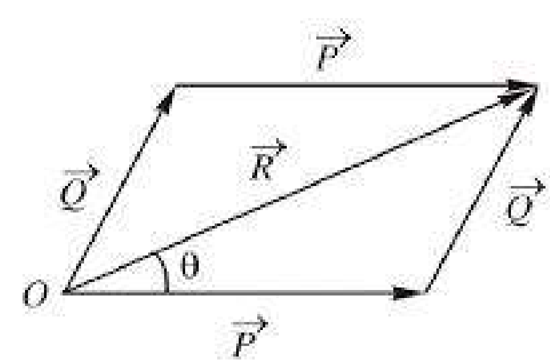


(*i*) (*ii*)

$$\vec{R} = \vec{P} + \vec{Q}$$

Hence the ***resultant vector*** *of two or more vectors is the single vector which produces the same effect as is produced by the individual vectors together.* The advantage of resultant vector is that we can replace a number of vectors by a single vector. This makes mathematical analysis a simple affair.

The method of vector addition described above is called *triangle law of vector addition.* An alternative graphical method for adding two vectors $\vec{P}$ and $\vec{Q}$ is shown in Fig. 8.7. In this construction, the tails of the vectors $\vec{P}$ and $\vec{Q}$ are together and the resultant vector $\vec{R}$ is the diagonal of the parallelogram formed with the vectors $\vec{P}$ and $\vec{Q}$ as the adjacent sides. This is called *parallelogram law of vector addition.*



(*i*) Triangle law of vector addition. *If two vectors acting simultaneously at a point are represented in magnitude and direction by the two sides of a triangle taken in the same order, then third or closing side of the triangle taken in the opposite order represents their resultant in magnitude and direction.*

The triangle law of vector addition is illustrated in Fig. 8.6(*ii*). Here vectors $\vec{P}$ and $\vec{Q}$ are represented in magnitude and direction by the two sides of a triangle taken in the same order. The third or closing side of the triangle taken in the opposite order represents their resultant $\vec{R}$ in magnitude and direction.

$$\therefore \qquad \vec{R} = \vec{P} + \vec{Q}$$

(*ii*) Parallelogram law of vector addition. *If two vectors acting simultaneously at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, then the diagonal of the parallelogram passing through that point represents their resultant in magnitude and direction.*

The parallelogram law of vector addition is shown in Fig. 8.7. Here vectors $\vec{P}$ and $\vec{Q}$ are represented in magnitude and direction by the adjacent sides of the parallelogram. Then diagonal of the parallelogram passing through point O represents their resultant $\vec{R}$ in magnitude and direction.

8.5. ADDITION OF MORE THAN TWO VECTORS GRAPHICALLY

The *tail-to-tip method can be used to add any number of vectors graphically. As stated earlier, in this method the vectors are placed tip to tail one at a time *i.e.*, tail of the second vector is placed on the tip of the first vector, tail of the third vector is placed on the tip of the second vector and so on. The magnitudes and directions of the vectors should not be changed as they are moved about. The resultant vector $\vec{R}$ is represented in magnitude and direction from the tail of the first vector to the tip of the last vector. In other words, $\vec{R}$ is the closing side of the figure (of vectors) taken in the opposite order.

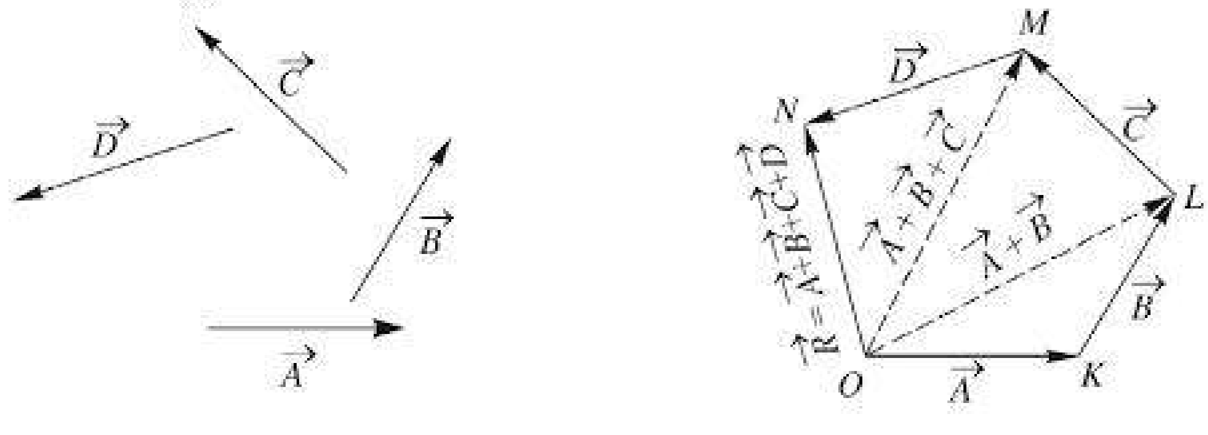


Fig. 8.8

Let us illustrate the tail-to-tip method of vector addition considering four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ [See Fig. 8.8 (*i*)]. Applying tail-to-tip method, these vectors are represented in magnitude and direction by the sides $\overrightarrow{OK}$, $\overrightarrow{KL}$, $\overrightarrow{LM}$ and $\overrightarrow{MN}$ of the regular polygon respectively as shown in Fig. 8.8 (*ii*). Then, the resultant $\vec{R}$ is represented in magnitude and direction from the tail of the first vector to the tip of the last vector. In other words, the closing side of the polygon taken in the opposite order represents the resultant $\vec{R}$ in magnitude and direction *i.e.*, $\vec{R} = \overrightarrow{ON}$. This provides

* Note that tail-to-tip method has been explained in Art. 8.4.

us a useful rule for adding more than two vectors graphically. This is called **polygon law of vector addition**. It may be stated as under:

If a number of vectors acting simultaneously at a point are represented in magnitude and direction by the sides of a polygon taken in the same order, then closing side of the polygon taken in opposite order represents the resultant in magnitude and direction.

Proof. A little thought will show that polygon law of vector addition is an extension of triangle law of vector addition. According to triangle law of vector addition, $\vec{OL}$ is the resultant of $\vec{A}$ and $\vec{B}$ *i.e.*,

$$\overrightarrow{OL} = \vec{A} + \vec{B}$$

Likewise, $\overrightarrow{OM}$ is the resultant of $\overrightarrow{OL}$ and $\vec{C}$ *i.e.*,

$$\overrightarrow{OM} = \overrightarrow{OL} + \vec{C} = \vec{A} + \vec{B} + \vec{C}$$

$$\therefore \quad \overrightarrow{OM} = \vec{A} + \vec{C} + \vec{B}$$

Again $\overrightarrow{ON}$ is the resultant of $\overrightarrow{OM}$ and $\vec{D}$.

$$\therefore \quad \overrightarrow{ON} = \overrightarrow{OM} + \vec{D} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

$$\therefore \quad \vec{R} = \overrightarrow{ON} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

8.6. ANALYTICAL METHOD OF VECTOR ADDITION

The graphical method of adding vectors takes considerable time if done accurately. We can find the magnitude and direction of the resultant (or sum) of two vectors mathematically by using triangle or parallelogram law of vector addition.

1. Triangle law of vector addition. Let two vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the sides $\overrightarrow{OA}$ and $\overrightarrow{AC}$ of a triangle OAC taken in the same order. Then according to triangle law of vector addition, the third or closing side of the triangle taken in the opposite order represents the resultant $\vec{R}\ (= \overrightarrow{OC})$ in magnitude and direction. Let $\vec{R}$ make an angle α with $\vec{P}$ *i.e.*, [See Fig. 8.9] $\angle AOC = \alpha$.

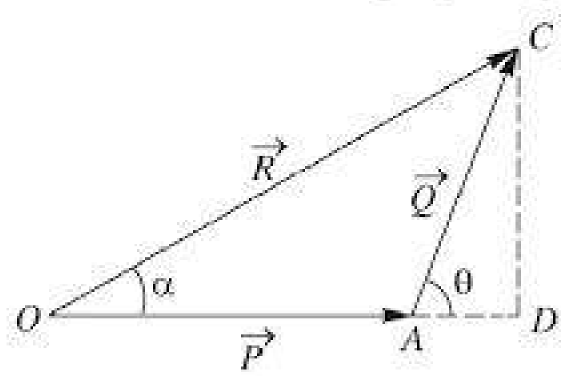


Fig. 8.9

(i) **Magnitude of Resultant.** From C, draw CD perpendicular to OA produced. Suppose $\angle CAD = \theta$. Note that θ is the angle between the two vectors. In the right angled triangle ODC,

$$(OC)^2 = (OD)^2 + (CD)^2 = (OA + AD)^2 + (CD)^2$$

[In right angled triangle ACD, $AD = Q\cos\theta$ and $CD = Q\sin\theta$]

$$\therefore \quad R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$= P^2 + Q^2(\sin^2\theta + \cos^2\theta) + 2PQ\cos\theta$$

$$= P^2 + Q^2 + 2PQ\cos\theta \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \quad \ldots (i)$$

(ii) Direction of resultant. From right angled triangle ODC,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\therefore \quad \tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta} \qquad ...(ii)$$

Thus, the magnitude of the resultant is given by eq. (*i*) and the direction of $\vec{R}$ with $\vec{P}$ is given by eq. (*ii*).

2. Parallelogram law of vector addition. Let two vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the parallelogam $OACB$ (See Fig. 8.10). Suppose the angle between the vectors is θ *i.e.*, $\angle AOB = \theta$. According to parallelogram law of vector addition, the diagonal $\overrightarrow{OC}$ represents the resultant $\vec{R}\,(=\overrightarrow{OC})$ in magnitude and direction. Suppose $\vec{R}$ makes angle α with $\vec{P}$ *i.e.*, $\angle AOC = \alpha$.

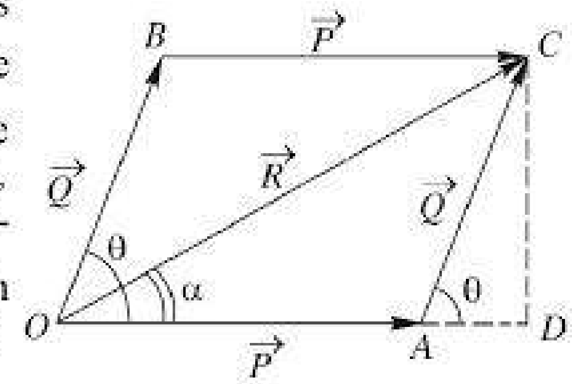


Fig. 8.10

(i) Magnitude of resultant. From C, draw CD perpendicular on OA produced. From geometry, $\angle DAC = \theta$. In right angled triangle ODC,

$$(OC)^2 = (OD)^2 + (CD)^2 = (OA+AD)^2 + (CD)^2$$

$$\therefore \quad R^2 = (P+Q\cos\theta)^2 + (Q\sin\theta)^2 \qquad [\because AD = Q\cos\theta;\ CD = Q\sin\theta]$$

$$= P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$

$$= P^2 + Q^2(\sin^2\theta + \cos^2\theta) + 2PQ\cos\theta$$

$$= P^2 + Q^2 + 2PQ\cos\theta$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \qquad ...(iii)$$

(ii) Direction of resultant. From right angled triangle ODC,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

$$\therefore \quad \tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta} \qquad ...(iv)$$

Thus, the magnitude of the resultant is given by eq. (*iii*) and the direction of $\vec{R}$ with $\vec{P}$ is given by eq. (*iv*).

Different cases.

(*a*) ***When the vectors act along the same direction*** *i.e.*, $\theta = 0°$.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 0°} = \sqrt{(P+Q)^2} \quad \therefore \quad R = P + Q$$

$$\tan\alpha = \frac{Q\sin 0°}{P+Q\cos 0°} = \frac{Q(0)}{P+Q(1)} = 0 \quad \therefore \quad \alpha = 0°$$

Therefore, the magnitude of the resultant of two vectors in the same direction is equal to the sum of the magnitudes of two vectors and the direction of the resultant is along the direction of $\vec{P}$ and $\vec{Q}$.

(*b*) ***When the vectors are at right angles*** *i.e.*, $\theta = 90°$.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 90°} = \sqrt{P^2 + Q^2}$$

$$\tan\alpha = \frac{Q\sin 90°}{P + Q\cos 90°} = \frac{Q(1)}{P + Q(0)} = \frac{Q}{P} \quad \therefore \quad \tan\alpha = \frac{Q}{P}$$

(*c*) ***When the vectors act along opposite directions*** *i.e.*, $\theta = 180°$.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 180°} = \sqrt{(P - Q)^2}$$

$$\therefore \quad R = P - Q$$

$$\tan\alpha = \frac{Q\sin 180°}{P + Q\cos 180°} = \frac{Q(0)}{P + Q(-1)} = 0$$

$$\therefore \quad \alpha = 0° \text{ or } 180°$$

Thus the magnitude of the resultant of two vectors acting in the opposite directions is equal to the difference in the magnitudes of the vectors. The resultant acts in the direction of the larger vector.

8.7. SOME PROPERTIES OF VECTORS

We now discuss some useful properties of vectors.

(*i*) Multiplication of a vector by a real number. *If we multiply a vector* $\vec{A}$ *by a real positive number n, we get the vector* $n\vec{A}$ *which has the same direction as* $\vec{A}$ *and magnitude nA i.e.,*

$$n\vec{A} = n(\vec{A})$$

Thus the magnitude of the vector becomes n times while its direction remains unchanged. For example, if we multiply vector $\vec{A}$ by 2, we get another vector $\vec{B}$ which has the same direction as that of $\vec{A}$ but twice the magnitude *i.e.*

$$\vec{B} = 2\vec{A}$$

If a vector is multiplied by a negative real number (i.e., –n), the magnitude of the vector becomes nA but direction is opposite to that of $\vec{A}$ *i.e.,*

$$-n(\vec{A}) = -n\vec{A}$$

For example, if we multiply vector $\vec{A}$ by –2, we get another vector $\vec{B}$ parallel to $\vec{A}$ but in opposite direction and of magnitude $2A$ *i.e.,*

$$\vec{B} = -2\vec{A}$$

(*ii*) Multiplication of a vector by a scalar. *If we multiply a vector* $\vec{A}$ *by a scalar S, the result is another vector with direction of* $\vec{A}$ *but magnitude SA. If S is negative,* $S\vec{A}$ *has a direction opposite to that of* $\vec{A}$.

If the quantities S and $\vec{A}$ both happen to have units, then vector $S\vec{A}$ is expressed in units indicated for such a product. For example, if $\vec{A}$ = 10 m/s north and S = 2 s, then $S\vec{A}$ = 10 m/s × 2 s = 20 m north.

(*iii*) Vector addition obeys commutative law. *According to this law, the resultant of the vectors remains the same in whatever order they may be added.*

In order to find the vector sum of $\vec{A}$ and $\vec{B}$, we can add (vectorially) either $\vec{A}$ to $\vec{B}$ or $\vec{B}$ to $\vec{A}$ (See Fig. 8.11). Vectors $\vec{A} + \vec{B}$ and $\vec{B} + \vec{A}$ are the same because they are parallel and have the same magnitude. Their order in the addition does not matter.

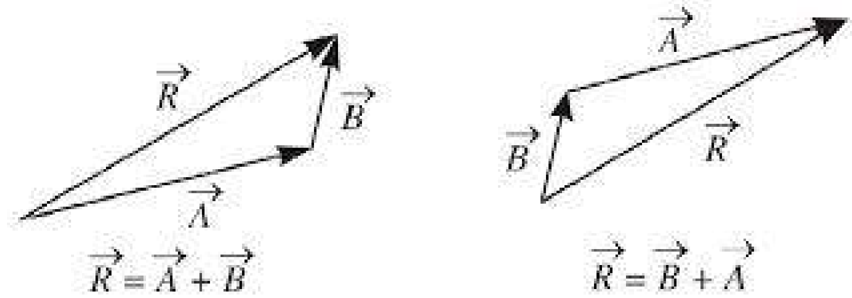


Fig. 8.11

$$\therefore \quad \vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

It is interesting to note that commutative law also holds for ordinary algebra because 3 + 2 = 2 + 3.

(iv) Vector addition obeys associative law. *According to this law, the resultant of the vectors remains the same in whatever grouping they may be added.*

Suppose we are to find the vector sum of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Then we can add $\vec{C}$ to the vector sum of $\vec{A} + \vec{B}$ or add $\vec{A}$ to the vector sum of $\vec{B} + \vec{C}$. Therefore, as shown in Fig. 8.12 :

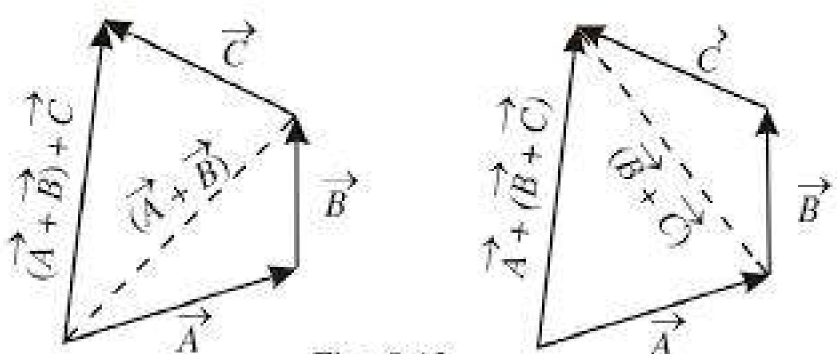


Fig. 8.12

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

This is quite logical. For the addition of any number of vectors, we can merely draw them in head-to-tail fashion. The sum of all vectors is then just a vector from the tail of the first vector to the head of the last vector. It is interesting to note that associative law also holds good for ordinary algebra because (2 + 3) + 4 = 2 + (3 + 4).

Example 8.1. The angle between two vectors of equal magnitude is 120°. Prove that the magnitude of their resultant is equal to either of them.

Solution. $P = Q\ ;\ \theta = 120°$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{P^2 + P^2 + 2P^2\cos 120°}$$

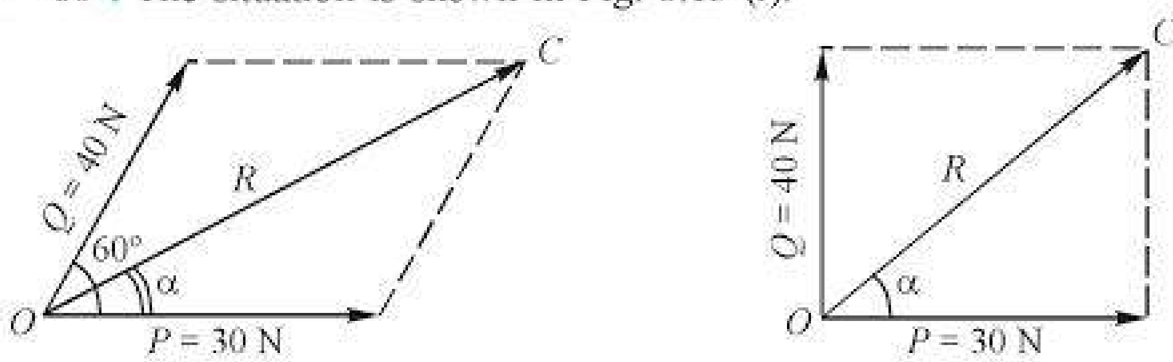
$$= \sqrt{2P^2 + 2P^2\left(\frac{-1}{2}\right)} = \sqrt{2P^2 - P^2} = P$$

$$\therefore \quad R = P$$

Example 8.2. Two forces of 30 N and 40 N are inclined to each other at an angle of 60°. Find their resultant. What will be the angle if the forces are inclined at right angles to each other?

Solution. $P = 30\ N\ ;\ Q = 40\ N$

When θ = 60°. The situation is shown in Fig. 8.13 (*i*).



(*i*) **Fig. 8.13** (*ii*)

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{(30)^2 + (40)^2 + 2 \times 30 \times 40\cos 60°} = \mathbf{60.83\ N}$$

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{40\sin 60°}{30 + 40\cos 60°} = 0.69 \quad \therefore \quad \theta = \tan^{-1} 0.69 = 34.6°$$

Therefore, the resultant has a magnitude of 60.83 N and makes an angle of 34.6° with 30 N force.

When θ = 90°. The situation is shown in Fig. 8.13 (*ii*).

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{P^2 + Q^2 + 2PQ\cos 90°}$$

$$= \sqrt{P^2 + Q^2} = \sqrt{(30)^2 + (40)^2} = 50\ \text{N}$$

$$\tan\alpha = \frac{Q}{P} = \frac{40}{30} = 1.33 \quad \therefore \quad \alpha = \tan^{-1} 1.33 = \mathbf{53.06°}$$

Therefore, the resultant has a magnitude of 50 N and makes an angle of 53.06° with 30 N force.

Example 8.3. The resultant of two equal forces acting at right angles to each other is 1414 N. Find the magnitude of each force.

Solution. $P = Q$; R = 1414 N ; $\theta = 90°$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = \sqrt{P^2 + P^2 + 2P^2\cos 90°} = \sqrt{2}\,P$$

$$\therefore \quad P = \frac{R}{\sqrt{2}} = \frac{1414}{\sqrt{2}} = 1000\ \text{N} \quad \therefore \quad P = Q = \mathbf{1000\ N}$$

Example 8.4. The magnitudes of two vectors are equal and the angle between them is θ. Show that their resultant divides angle θ equally.

Solution. $P = Q$; $\theta = \theta$

$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} = \frac{P\sin\theta}{P + P\cos\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \quad \therefore \quad \alpha = \frac{\theta}{2}$$

We can also find the magnitude of the resultant.

$$R^2 = P^2 + P^2 + 2P^2\cos\theta = 2P^2 + 2P^2\cos\theta = 2P^2(1 + \cos\theta)$$

$$= 2P^2\left[1 + \left(2\cos^2\frac{\theta}{2} - 1\right)\right] = 4P^2\cos^2\frac{\theta}{2}$$

$$\therefore \quad R = 2P\cos\frac{\theta}{2}$$

These results are very important.

Example 8.5. The sum of the magnitudes of two forces acting at a point is 18 N and the magnitude of the resultant is 12 N. If the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?

Solution. Fig. 8.14 shows the conditions of the problem. Here $\vec{P}$ and $\vec{Q}$ are the two forces and $\vec{R}$ is their resultant. It is assumed that $\vec{P}$ is the smaller force.

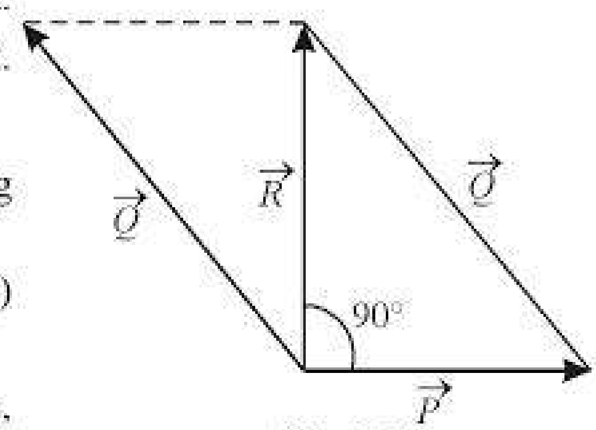


Fig. 8.14

It is given that $P + Q = 18$ N and $R = 12$ N. Referring to Fig. 8.14, we have,

$$R^2 = Q^2 - P^2 \quad \text{or} \quad R^2 = (Q + P)(Q - P)$$

$$\therefore \quad 144 = 18(Q - P) \quad \text{or} \quad Q - P = 8 \text{ N}$$

Now $P + Q = 18$ N and $Q - P = 8$ N. Therefore, $Q =$ **13 N** and $P =$ **5 N.**

Example 8.6. A particle has a displacement of 12 m toward east and 5 m toward north and then 6 m vertically upward. Find the magnitude of the resultant displacement.

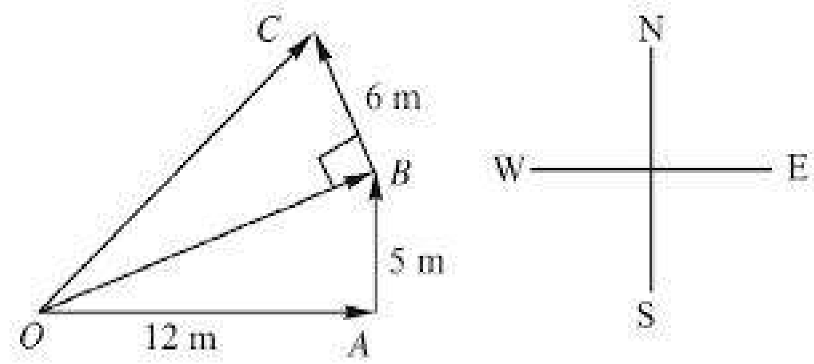


Fig. 8.15

Solution. Fig. 8.15 shows the conditions of the problem. $\overrightarrow{OB}$ is the resultant of $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

$$\therefore \quad OB = \sqrt{(OA)^2 + (AB)^2} = \sqrt{(12)^2 + (5)^2} = 13 \text{ m}$$

Again $\overrightarrow{OC}$ is the resultant of $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

$$\therefore \quad OC = \sqrt{(OB)^2 + (BC)^2} = \sqrt{(13)^2 + (6)^2} = \sqrt{205} = \mathbf{14.32 \text{ m}}$$

Example 8.7. A person moves 30 m north, then 20 m east and finally $30\sqrt{2}$ m south-west. What is the displacement from the original position?

Solution. Figure 8.16 shows the conditions of the problem. Resultant displacement is given by ;

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$$

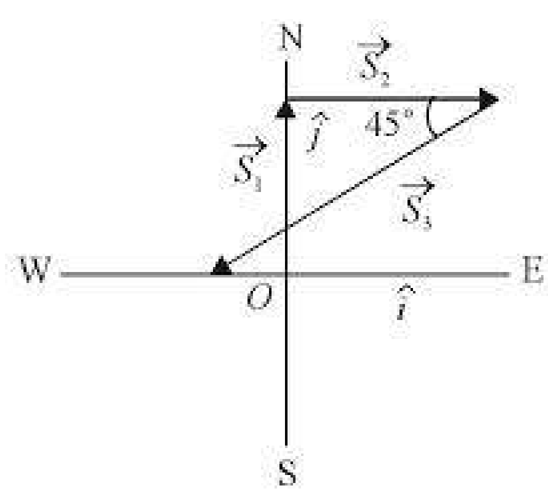


Fig. 8.16

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along east (E) and north (N) respectively. Then,

$$\vec{S}_1 = 30\,\hat{j}\,;\ \vec{S}_2 = 20\hat{i}\,;$$

$$\vec{S}_3 = -30\sqrt{2}(\cos 45°\hat{i} + \sin 45°\hat{j}) = -30\hat{i} - 30\hat{j}$$

$$\therefore \quad \vec{S} = (30\hat{j}) + (20\hat{i}) + (-30\hat{i} - 30\hat{j}) = -10\hat{i}$$

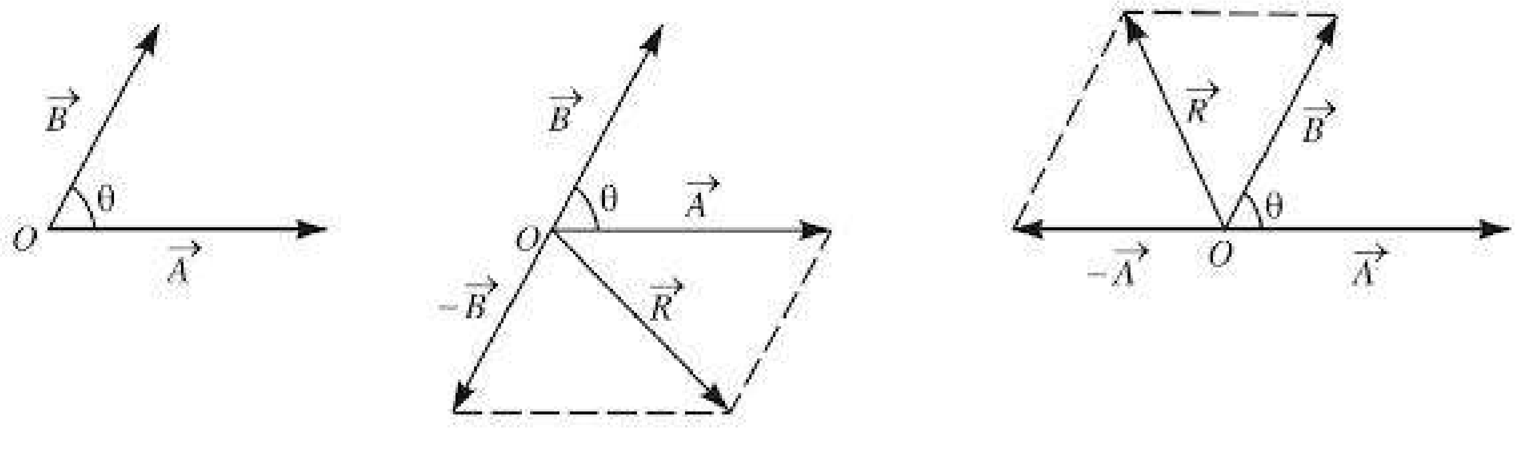
Therefore, the resultant displacement is **10 m west.**

8.8. SUBTRACTION OF VECTORS

In order to subtract vector $\vec{B}$ from vector $\vec{A}$, first reverse the direction of $\vec{B}$, thus producing $-\vec{B}$. Then add $\vec{A}$ and $(-\vec{B})$ by parallelogram law of vector addition.

$$\therefore \quad \vec{R} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

Consider two vectors $\vec{A}$ and $\vec{B}$ of the same kind and inclined to each other at an angle θ as shown in Fig. 8.17. (*i*).



(*i*) (*ii*) (*iii*)

Fig. 8.17

(*i*) In order to find $\vec{A}-\vec{B}$, reverse the direction of $\vec{B}$, thus producing $-\vec{B}$ as shown in Fig. 8.17 (*ii*). Then find the sum of $\vec{A}$ and $-\vec{B}$ by parallelogram law of vector addition. The required difference is $\vec{R}$ and is shown in Fig. 8.17 (*ii*).

$$\therefore \qquad \vec{R} = \vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

(*ii*) In order to find $\vec{B}-\vec{A}$, reverse the direction of $\vec{A}$, thus producing $-\vec{A}$ as shown in Fig. 8.17(*iii*). Then find the sum of $\vec{B}$ and $-\vec{A}$. The required difference is $\vec{R}$ and is shown in Fig. 8.17 (*iii*).

$$\therefore \qquad \vec{R} = \vec{B} + (-\vec{A}) = \vec{B} - \vec{A}$$

8.9. RELATIVE VELOCITY IN TWO DIMENSIONS

Two bodies A and B moving in a plane (two dimensional motion) may not have their motions along the same line. Consequently, their velocities $\vec{v}_A$ and $\vec{v}_B$ will be inclined at some angle as shown in Fig. 8.18.

(*i*) In order to find the relative velocity of body A *w.r.t.* to body B $(\vec{v}_{AB})$, we superimpose a velocity $-\vec{v}_B$ on both the bodies. As a result, the body B is brought to rest. Then the resultant velocity of the body A is $\vec{v}_A - \vec{v}_B$ and is equal to $\vec{v}_{AB}$.

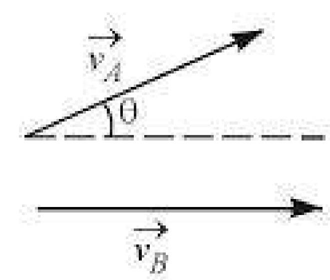


Fig. 8.18

$$\therefore \qquad \vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$$

Therefore, in order to find $\vec{v}_{AB}$, reverse the direction of $\vec{v}_B$ and find its sum with $\vec{v}_A$ by parallelogram law of vector addition.

(*ii*) Similarly, $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$

Again parallelogram law of vector addition can be used to find the sum of $\vec{v}_B$ and $-\vec{v}_A$.

8.10. MAGNITUDE AND DIRECTION OF RELATIVE VELOCITY

Consider two bodies A and B moving with velocities $\vec{v}_A$ and $\vec{v}_B$ respectively inclined at an angle θ.

(*i*) Relative velocity of *A w.r.t. B*. As discussed above, relative velocity of A *w.r.t.* B is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$$

In order to find $\vec{v}_{AB}$, we find the sum of $\vec{v}_A$ and $-\vec{v}_B$ by parallelogram law of vector addition. This is illustrated in Fig. 8.19 (*i*) where the vector sum of $\vec{v}_A$ and $-\vec{v}_B$ is $\vec{v}_{AB} (= \overrightarrow{OC})$.

Magnitude of $\vec{v}_{AB}$ is

$$v_{AB} = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180° - \theta)}$$

or $$v_{AB} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

The angle α which $\vec{v}_{AB}$ makes with $-\vec{v}_B$ is

$$\tan\alpha = \frac{v_A \sin(180° - \theta)}{v_B + v_A \cos(180° - \theta)} \quad \text{or} \quad \tan\alpha = \frac{v_A \sin\theta}{v_B - v_A \cos\theta}$$

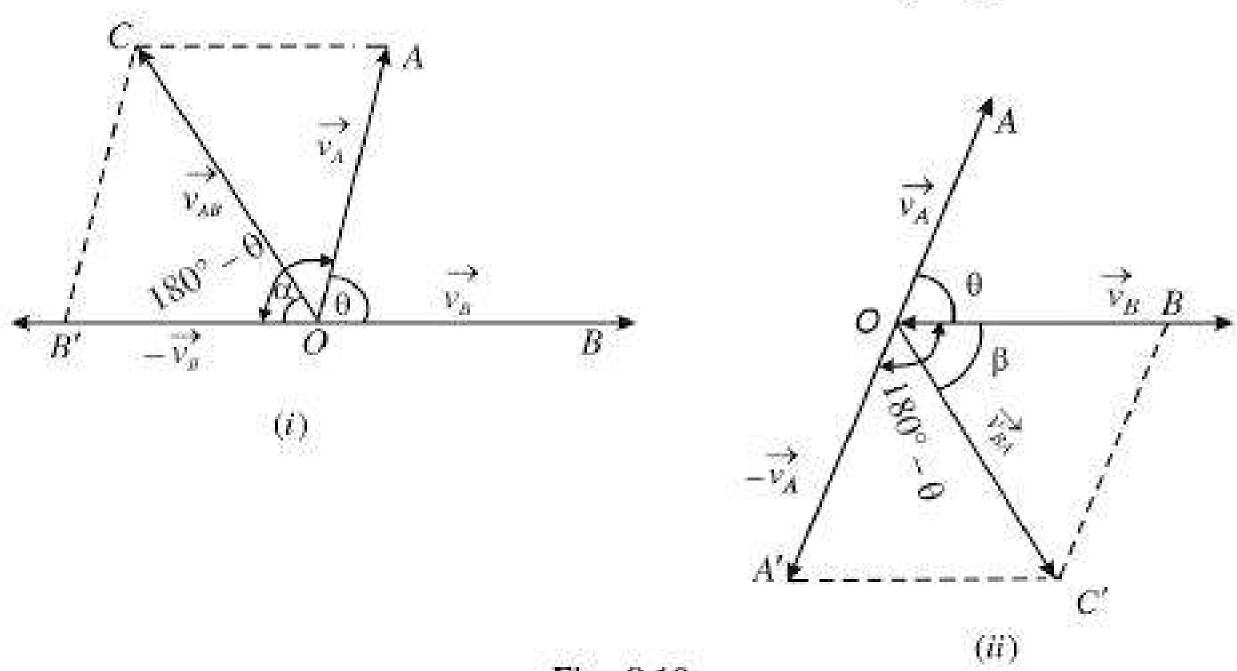


Fig. 8.19

(*ii*) Relative velocity of *B* w.r.t. *A*. The relative velocity of *B* w.r.t. *A* is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

In order to find $\vec{v}_{BA}$, we find the sum of $\vec{v}_B$ and $-\vec{v}_A$ by parallelogram law of vector addition. This is illustrated in Fig. 8.19 (*ii*) where the vector sum of $\vec{v}_B$ and $-\vec{v}_A$ is $\vec{v}_{BA}$ $(= \overrightarrow{OC'})$.

Magnitude of $\vec{v}_{BA}$ is

$$v_{BA} = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180° - \theta)}$$

or $$v_{BA} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

The angle β which $\vec{v}_{BA}$ makes with $\vec{v}_B$ is

$$\tan\beta = \frac{v_A \sin(180° - \theta)}{v_B + v_A \cos(180° - \theta)}$$

or $$\tan\beta = \frac{v_A \sin\theta}{v_B - v_A \cos\theta}$$

8.11. APPLICATIONS OF RELATIVE VELOCITY

We shall discuss a few applications of relative velocity in two dimensions.

(i) Boat to cross the river along shortest path. Fig. 8.20 shows river flowing from right to left. Point A is the starting point of the boat and point B is directly opposite to point A on the other side of the river. It is desired that boat should follow the shortest path AB. The river flow will carry the boat downstream as the boat crosses the river. In order that the boat follows the path AB, it should be rowed upstream making an angle θ with AB as shown in Fig. 8.20.

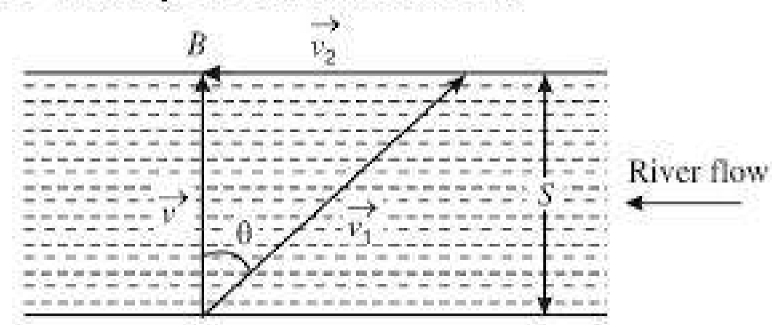


Fig. 8.20

Let $\overrightarrow{v_{BS}} = \vec{v_1}$ = Velocity of boat *w.r.t.* shore

$\overrightarrow{v_{WS}} = \vec{v_2}$ = Velocity of water *w.r.t.* shore

$\overrightarrow{v_{BW}} = \vec{v}$ = Velocity of boat *w.r.t.* water

$$\therefore \quad \overrightarrow{v_{BW}} = \overrightarrow{v_{BS}} + \overrightarrow{v_{SW}} = \overrightarrow{v_{BS}} - \overrightarrow{v_{WS}}$$

or
$$\vec{v} = \vec{v_1} - \vec{v_2}$$

The magnitude of $\vec{v}\ (= \overrightarrow{v_{BW}})$ is given by ;

$$v = \sqrt{v_1^2 - v_2^2}$$

Now the boat moves across AB with a velocity v.

$\therefore$ Time of crossing of boat, $t = \dfrac{AB}{v} = \dfrac{S}{v}$

Angle θ is given by ; $\sin\theta = v_2/v_1$

Note that angle θ is determined by the magnitudes of $\vec{v_2}$ and $\vec{v_1}$.

(ii) Boat to cross the river in shortest time. In order that the boat crosses the river in the shortest time, the river velocity $\vec{v_2}$ should help the boat velocity $\vec{v_1}$. Therefore, the boat should be rowed along AB. Now the resultant velocity $\vec{v}\ (= \overrightarrow{v_{BW}})$ is given by (See Fig. 8.21) ;

$$\vec{v} = \vec{v_1} + \vec{v_2}$$

Magnitude of $\vec{v}$, $v = \sqrt{v_1^2 + v_2^2}$; $\tan\theta = \dfrac{v_2}{v_1}$

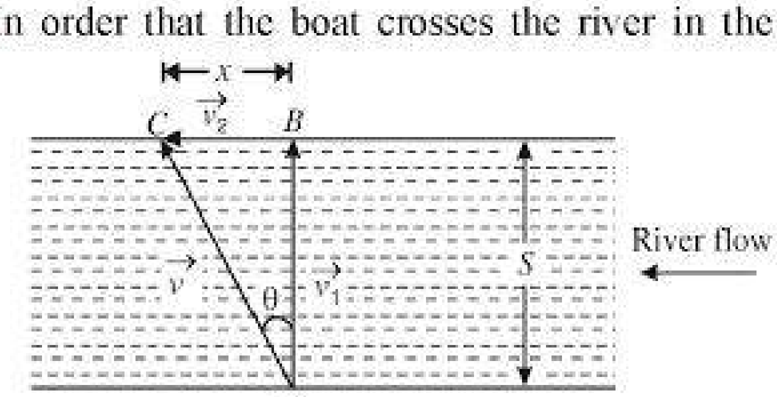


Fig. 8.21

Now the boat moves with velocity v along AC and will reach point C instead of point B. If $BC = x$, then,

$$AC = \sqrt{x^2 + S^2} \quad \text{where } S = \text{width of the river}$$

$\therefore$ Time of crossing of boat is

$$t = \frac{AC}{v} = \sqrt{\frac{x^2 + S^2}{v_1^2 + v_2^2}}$$

Now
$$\tan\theta = \frac{v_2}{v_1} = \frac{x}{S} \quad \therefore \quad x = \frac{S v_2}{v_1}$$

(iii) Relative velocity of rain w.r.t. man. Consider a man walking east with velocity $\vec{v}_m$. Let the rain be falling vertically downward with velocity $\vec{v}_r$. In order to find the relative velocity of rain w.r.t. man $(\vec{v}_{rm})$, impress a velocity $-\vec{v}_m$ on rain as well as man. Now $\vec{v}_{rm}$ is the resultant of $-\vec{v}_m$ and $\vec{v}_r$ as shown in Fig. 8.22.

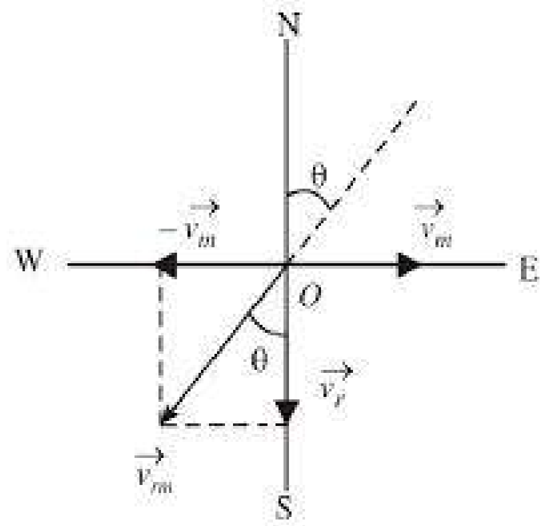


Fig. 8.22

Magnitude of $\vec{v}_{rm}$ is $v_{rm} = \sqrt{v_r^2 + v_m^2 + 2v_r v_m \cos 90°}$

or $v_{rm} = \sqrt{v_r^2 + v_m^2}$

If θ is the angle which $\vec{v}_{rm}$ makes with the vertical direction, then,

$$\tan\theta = \frac{v_m}{v_r} \quad \text{... (i)}$$

In order to save himself from the rain, the man should hold his umbrella in the direction of relative velocity of rain w.r.t. man $(\vec{v}_{rm})$. In other words, the umbrella should be held at an angle θ with the vertical direction given by eq. (*i*) above.

Example 8.8. A river 1 km wide is flowing at 3 km/h. A swimmer whose velocity in still water is 4 km/h can swim only for 15 minutes. In what direction should he strike out in order to reach the other bank? What is total distance covered?

Solution. Fig. 8.23 shows the conditions of the problem. The point *O* represents the starting position of the swimmer. At this point, man's velocity $\overrightarrow{OB}$ = 4 km/h and that of water current $\overrightarrow{OA}$ = 3 km/h. The resultant velocity of man will be $\overrightarrow{OB'}$ and he will reach point *P* on the other side of the bank.

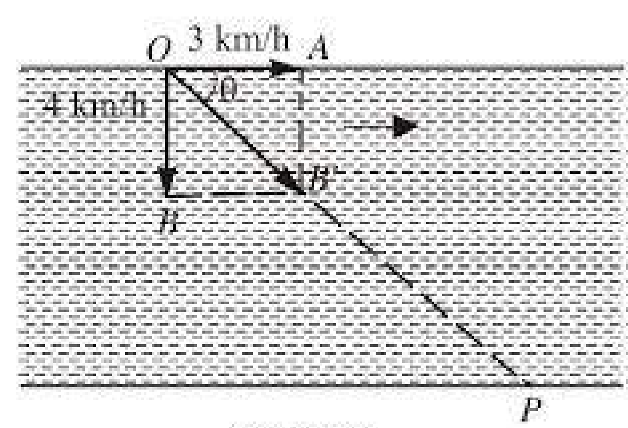


Fig. 8.23

$\therefore \quad \tan\theta = \frac{4}{3}$ or $\theta = \tan^{-1}\frac{4}{3} =$ **53.1°**

Magnitude of resultant velocity is

$$OB' = \sqrt{(OA)^2 + (OB)^2} = \sqrt{(3)^2 + (4)^2} = 5 \text{ km/h}$$

$\therefore$ Distance covered by the swimmer = $OB' \times \text{time} = 5 \times \frac{15}{60}$ = **1.25 km** along OB'

Example 8.9. A boatman can row with a speed of 10 km/h in still water. If the river flows steadily at 5 km/hr, in which direction should the boatman row in order to reach a point on the other bank directly opposite to the point from where he started? The width of the river is 2 km.

Solution. Suppose the river is flowing from left to right and O is the starting point. The boatman wants to reach point P directly opposite to the point O (See Fig. 8.24). If he starts along OP, he will be swayed away due to flow of water and may reach point P''. Therefore, in order to reach point P, he should start along OP' (upstream), making an angle θ with OP. The resultant velocity is now $\overrightarrow{OB'}$. $\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{OB}$

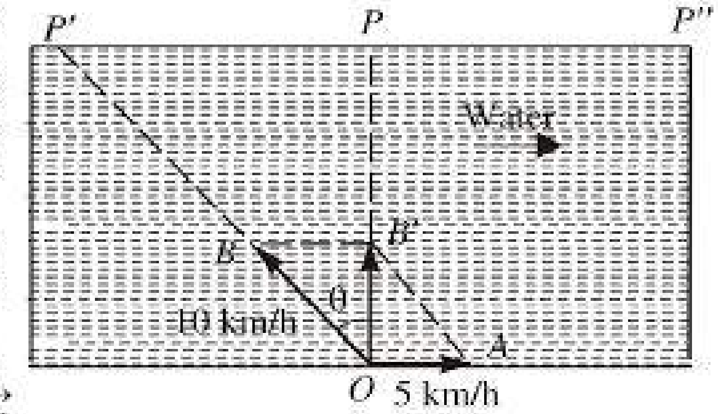


Fig. 8.24

In right angled triangle $OB'B$, $\sin\theta = \frac{BB'}{OB} = \frac{5}{10} = 0.5 \;\therefore\; \theta = 30°$

$\therefore$ Angle with the bank of river = $90° + \theta$ = $90° + 30°$ = **120°**

Therefore, the boatman should row making an angle of 120° with the bank of the river.

Example 8.10. A man is going due east with a velocity of 3 km/h. Rain falls vertically downward with a speed of 10 km/h. Find the angle at which he should hold his umbrella so as to save himself from rain.

Solution. Fig. 8.25 shows the conditions of the problem.

$\vec{v}_m$ = velocity of man = 3 km/h due east

$\vec{v}_r$ = velocity of rain = 10 km/h vertically downward

$\vec{v}_{rm}$ = velocity of rain *w.r.t.* man

In order to save himself from the rain, the man should hold his umbrella in the direction of relative velocity of rain w.r.t. man (*i.e.* $\vec{v}_{rm}$).

Now $\qquad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = \vec{v}_r + (-\vec{v}_m)$

In order to find $\vec{v}_{rm}$, reverse the direction of $\vec{v}_m$, thus producing $-\vec{v}_m$. Now find the sum of $\vec{v}_r$ and $-\vec{v}_m$ by parallelogram law of vector addition as shown in Fig. 8.25. The umbrella should be held at angle θ with the vertical where,

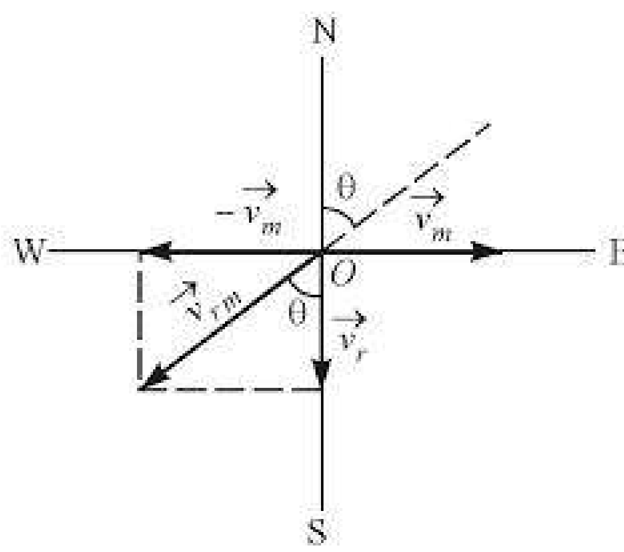


Fig. 8.25

$$\tan\theta = \frac{v_m}{v_r} = \frac{3}{10} = 0.3 \;\therefore\; \theta = \mathbf{16.7°}$$

Example 8.11. A boy is moving with a velocity of 3 km/h along east and the rain is falling vertically with a velocity of 4 km/h. Calculate the velocity of rain *w.r.t.* the boy.

Solution. Suppose $\vec{v}$ is the velocity of the boy and $\vec{u}$ that of rain. Then velocity of rain *w.r.t.* the boy is

$$\vec{V} = \vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

Taking x-axis along east and z-axis vertically upward (See Fig. 8.26), we have,

$$\vec{u} = -4\hat{k} \;;\; \vec{v} = 3\hat{i}$$

$\therefore \qquad \vec{V} = (-4\hat{k}) + (-3\hat{i}) = -4\hat{k} - 3\hat{i}$

Magnitude of $\vec{V}$ is given by ;

$$|\vec{V}| = \sqrt{(-4)^2 + (-3)^2} = \textbf{5 km/h}$$

If θ is the angle made by $\vec{V}$ with the vertical, then,

$$\tan\theta = +\frac{3}{4} = 0.75 \quad \therefore \ \theta = 37°$$

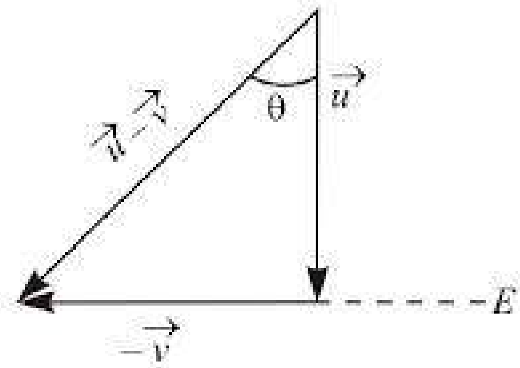


Fig. 8.26

Example 8.12. ***ABCD* is a parallelogram and $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are its diagonals. Prove that :**

(*i*) $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$ (*ii*) $\overrightarrow{AC} - \overrightarrow{BD} = 2\overrightarrow{AB}$

Solution. Fig. 8.27 shows the conditions of the problem.

(*i*) Applying triangle law of vectors to ΔABC, we have,

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \qquad ... (i)$$

Applying triangle law of vectors to ΔBCD, we have,

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{AB} \qquad ... (ii) \qquad (\because \ \overrightarrow{CD} = -\overrightarrow{AB})$$

Adding eqs. (*i*) and (*ii*), we have,

$$\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{BC} - \overrightarrow{AB}$$

$$\therefore \ \overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}$$

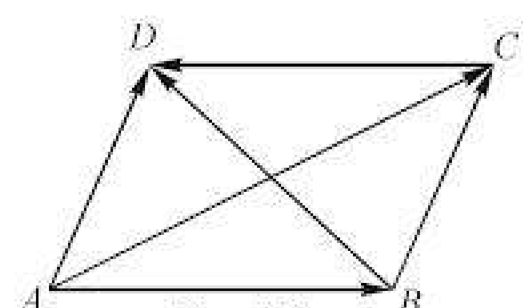


Fig. 8.27

(*ii*) Subtracting eq. (*ii*) from eq. (*i*), we have,

$$\overrightarrow{AC} - \overrightarrow{BD} = (\overrightarrow{AB} + \overrightarrow{BC}) - (\overrightarrow{BC} - \overrightarrow{AB})$$

$$= \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{BC} + \overrightarrow{AB}$$

$$\therefore \ \overrightarrow{AC} - \overrightarrow{BD} = 2\overrightarrow{AB}$$

PROBLEMS FOR PRACTICE

1. A boy goes 100 m north and then 70 m east. Find the displacement of the boy from the starting point. **[122 m]**
2. Two forces of 5 N and 10 N acting at a point have an angle of 120° between them. Find the magnitude and direction of the resultant force. **[8.66 N ; 30° with 10 N force]**
3. Two equal forces F have a resultant of 1.5 F. Find the angle between the forces. **[82.82°]**
4. A boat's speed in still water is 20 km/h. If the boat is to travel directly across a river whose current has a speed of 12 km/h, at what upstream angle must the boat head? **[126.9° with river bank]**
5. A swimmer is capable of swimming at 1.65 ms^{-1} in still water. (*i*) If she swims directly across a 180 m wide river whose current is 0.85 m/s, how far downstream (from a point opposite her starting point) will she land? (*ii*) How long will it take her to reach the other side? **[(*i*) 93 m (*ii*) 110 s]**

8.12. RESOLUTION OF A VECTOR IN A PLANE

When we add two vectors such as $\vec{A}$ and $\vec{B}$, we can replace the separate vectors by a single equivalent resultant vector, $\vec{R} = \vec{A} + \vec{B}$. We can do this in reverse *i.e.,* we can replace a single vector by any two (or more) vectors whose sum gives us back the original vector. This is called resolution of a vector.

The process of splitting a single vector into two or more vectors in different directions in a plane such that their sum gives back the original vector is called ***resolution of a vector.***

The vectors into which the given vector $\vec{R}$ is resolved (or splitted) are called the **vector components** of $\vec{R}$. Fig. 8.28 shows the vector $\vec{R}$ resolved into two non-parallel vectors $\vec{A}$ and $\vec{B}$ such that :

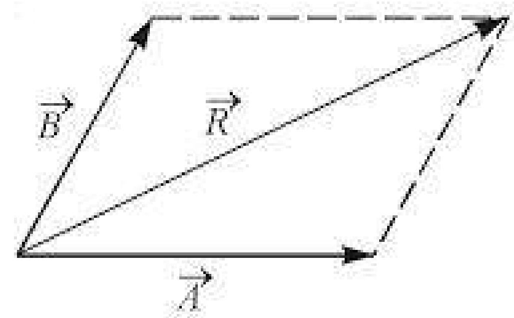


Fig. 8.28

$$\vec{R} = \vec{A} + \vec{B}$$

Therefore, $\vec{A}$ and $\vec{B}$ are the vector components of $\vec{R}$. As any number of vectors can be combined to give a single equivalent vector, therefore, a vector can be resolved into any number of vector components. The resolution of a vector into its components often simplifies many problems related to vectors.

8.13. RECTANGULAR COMPONENTS OF A VECTOR IN A PLANE

When a vector in a plane is resolved (i.e., splitted) into two components at right angles to each other, the component vectors are called ***rectangular components*** *of the vector.*

Fig. 8.29(*i*) shows a vector $\vec{A}$ in the *X*-*Y* plane. It is resolved into two rectangular components $\vec{A}_x$ and $\vec{A}_y$ (along *X*-axis and *Y*-axis).

$$\therefore \qquad \vec{A} = \vec{A}_x + \vec{A}_y$$

The vector components $\vec{A}_x$ and $\vec{A}_y$ are the rectangular components of vector $\vec{A}$. Therefore, we can replace vector $\vec{A}$ by $\vec{A}_x$ and $\vec{A}_y$. From the geometry of Fig. 8.29(*i*), we see that magnitudes of components $\vec{A}_x$ and $\vec{A}_y$ are related to the magnitude of $\vec{A}$ by ;

$$A_x = A\cos\theta\ ;\ A_y = A\sin\theta$$

Note that A_x and A_y are not vectors ; these are *x*-component and *y*-component respectively of vector $\vec{A}$. Therefore, we can treat them as *algebraic* quantities. Thus, we can specify vector $\vec{A}$ in a plane by scalar components A_x (= $A\cos\theta$) and A_y (= $A\sin\theta$).

Magnitude of $\vec{A}$, $A = \sqrt{A_x^2 + A_y^2} = \sqrt{(A\cos\theta)^2 + (A\sin\theta)^2}$

Direction of $\vec{A}$ with *X*-axis, $\tan\theta = \dfrac{A_y}{A_x}$

In short, we require two quantities to specify a vector such as $\vec{A}$ *viz.* either A and θ or A_x and A_y.

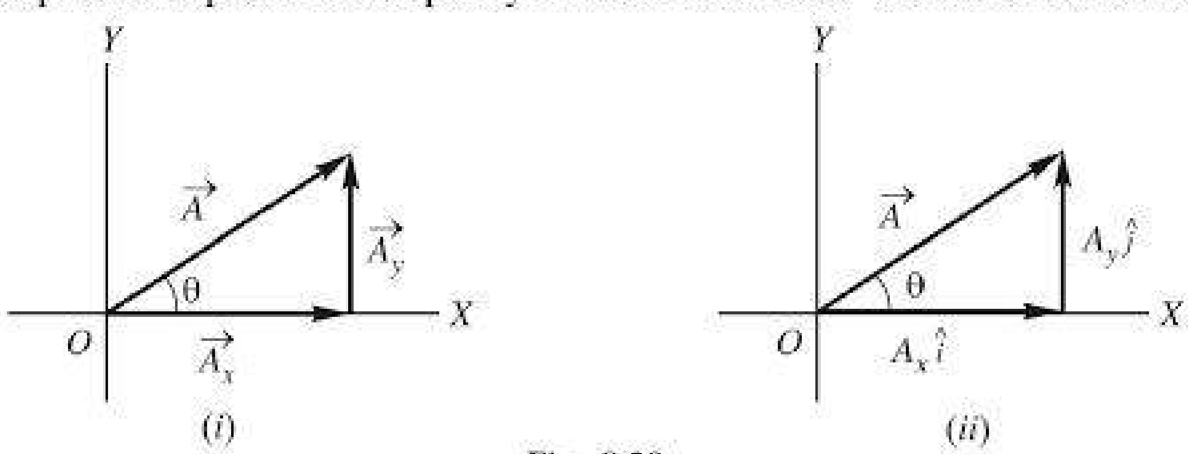


Fig. 8.29

We can express the vectors $\vec{A_x}$ and $\vec{A_y}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$ [See Fig. 8.29 (*ii*)].

$$\vec{A_x} = A_x\,\hat{i}\ ;\ \vec{A_y} = A_y\,\hat{j}$$

$$\therefore \qquad \vec{A} = A_x\,\hat{i} + A_y\,\hat{j}$$

Note that A_x (= A cos θ) is not a vector ; $A_x\hat{i}$ is a vector. Similarly, A_y (= A sin θ) is not a vector ; $A_y\hat{j}$ is a vector.

8.14. VECTOR ADDITION BY RECTANGULAR COMPONENTS

Adding vectors graphically is not very accurate and is not useful for vectors in three dimensions. A far more powerful and accurate method for adding vectors is to resolve the vectors into their rectangular components.

Fig. 8.30 shows two vectors $\vec{A}$ and $\vec{B}$ in the XY plane. By triangle law of vectors, their vector sum is $\vec{R}$. Note that vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ are resolved into rectangular components. It is easy to see the component R_x of $\vec{R}$ along X-axis is equal to the algebraic sum of the components of the vectors $\vec{A}$ and $\vec{B}$ along X-axis *i.e.*,

$$R_x = A_x + B_x$$

Similarly, the component R_y of $\vec{R}$ along Y-axis is equal to the sum (algebraic) of the components of vector $\vec{A}$ and $\vec{B}$ along Y-axis *i.e.*,

$$R_y = A_y + B_y$$

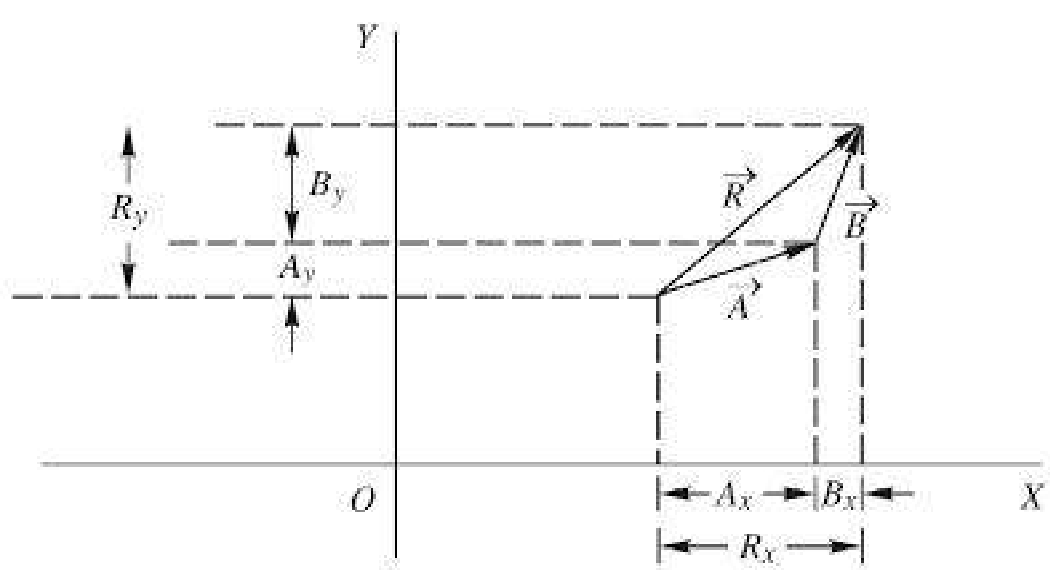


Fig. 8.30

The magnitude and direction of the resultant is given by ;

$$R = \sqrt{R_x^2 + R_y^2}\ ;\ \tan\theta = \frac{R_y}{R_x}$$

Note that angle θ (not shown) is the angle that $\vec{R}$ makes with the positive X-direction.

Example 8.13. Find the resultant of the following forces acting simultaneously at a point.

(*i*) a force of 50 N acting along *OX* axis

(*ii*) a force of 40 N acting at an angle of 60° to *OX* axis

(*iii*) a force of 60 N acting 330° with *OX* axis.

Solution. Fig. 8.31 (*i*) shows the conditions of the problem. Let R be the magnitude of the resultant of the forces. Resolving the forces into rectangular components, we have,

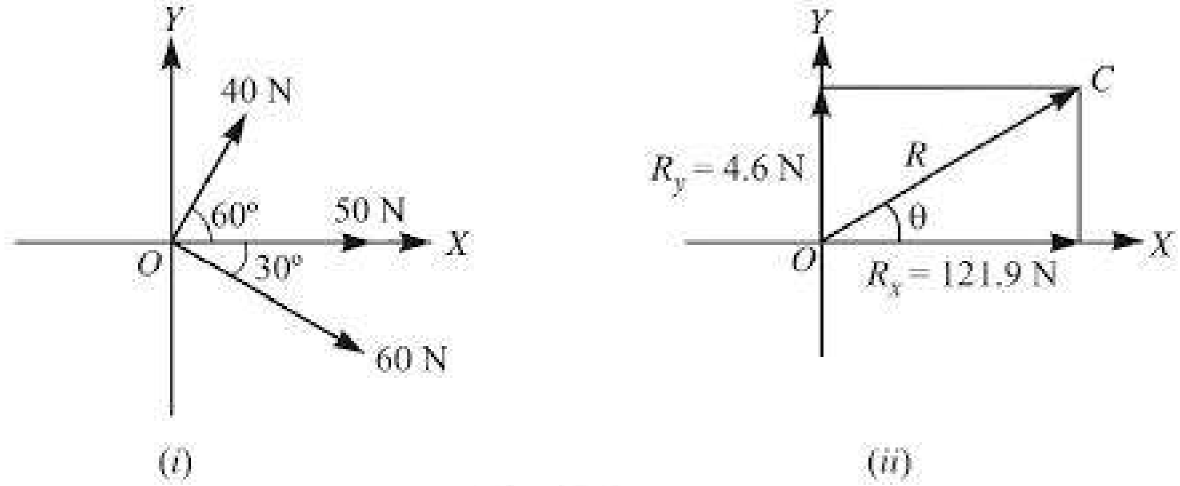


Fig. 8.31

The x-component of R, R_x = *Algebraic sum of x-components of forces

$= 50 \cos 0° + 40 \cos 60° + 60 \cos 30°$

$= 50 + 20 + 51.9 = 121.9$ N

The y-component of R, R_y = Algebraic sum of y-components of forces

$= 50 \sin 0° + 40 \sin 60° - 60 \sin 30°$

$= 0 + 34.6 - 30 = 4.6$ N

As shown in Fig. 8.31 (*ii*), R (= OC) represents the magnitude of the resultant.

$$\therefore \quad R = \sqrt{R_x^2 + R_y^2} = \sqrt{(121.9)^2 + (4.6)^2} = \mathbf{122\ N}$$

$$\tan\theta = \frac{R_y}{R_x} = 0.0377 \quad \therefore \quad \theta = \tan^{-1} 0.0377 = 2.2°$$

Note. Following the usual convention, the angle θ is measured counterclockwise from the positive X-axis.

8.15. RECTANGULAR COMPONENTS IN THREE DIMENSIONS

So far we have considered the rectangular components of a vector in a plane. The treatment can be extended to three-dimensional space in which case a vector will have three rectangular components. Consider a vector $\vec{A}$ in space. It has three rectangular components $\vec{A}_x$, $\vec{A}_y$ and $\vec{A}_z$ as shown in Fig. 8.32.

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

Magnitude of $\vec{A}$, **$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Note that $\vec{A}_x$ is the x-component of $\vec{A}$; $\vec{A}_y$ is the y-component of $\vec{A}$ and $\vec{A}_z$ is the z-component of $\vec{A}$.

---

* An algebraic sum is one in which the sign of quantity is to be taken into account. The component along OX axis is positive while in opposite direction, it is negative. Similarly, component along OY is positive and that in the opposite direction is negative.

** Note that A_x and A_y are perpendicular to each other and their resultant is $R' = \sqrt{A_x^2 + A_y^2}$. Now A_z is $\perp$ R'. Therefore, magnitude of $A = \sqrt{R'^2 + A_z^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

We can also express vector $\vec{A}$ in terms of unit vectors as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Note that A_x is not a vector ; $A_x\hat{i}$ is a vector. Same is true for other two components.

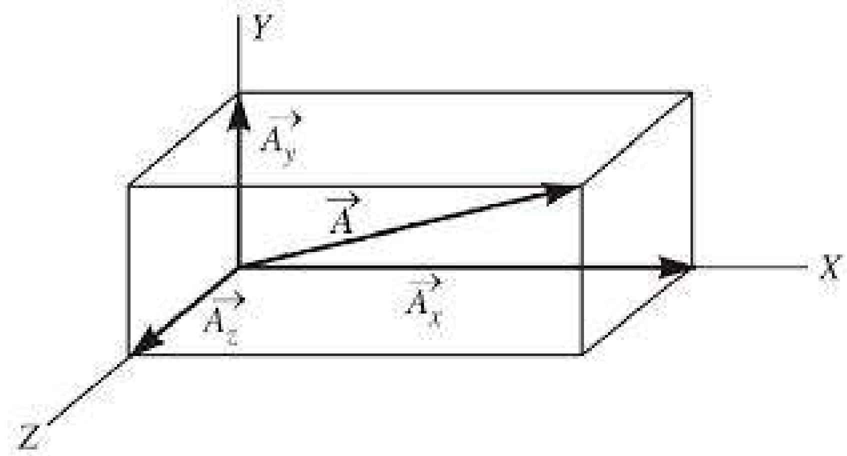


Fig. 8.32

Addition of vectors in three dimensions. To add vectors in three dimensions, each vector is resolved into three rectangular components and the treatment is carried out in a similar way as in the two dimensional case. Suppose that there are three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ in space and their sum is $\vec{R}$.

$$\therefore \qquad \vec{R} = \vec{A} + \vec{B} + \vec{C}$$

The three corresponding algebraic equations are :

$$R_x = A_x + B_x + C_x \;;\; R_y = A_y + B_y + C_y \;;\; R_z = A_z + B_z + C_z$$

Note the economy, generality and elegance of vector algebra. A single vector equation represents three component equations.

Direction cosines of vector. If α, β and γ are the angles which vector $\vec{A}$ makes with X, Y and Z axes respectively, then,

$$\cos\alpha = \frac{A_x}{A} \quad \text{or} \quad A_x = A\cos\alpha$$

$$\cos\beta = \frac{A_y}{A} \quad \text{or} \quad A_y = A\cos\beta$$

$$\cos\gamma = \frac{A_z}{A} \quad \text{or} \quad A_z = A\cos\gamma$$

Here $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called **direction cosines** of the vector $\vec{A}$.

Now $\qquad A^2 = A_x^2 + A_y^2 + A_z^2$

or $\qquad A^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$

or $\qquad 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

Therefore, the sum of the squares of the direction cosines of a vector is always unity.

Example 8.14. A force is represented by ;

$$\vec{F} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ newton.}$$

What is the magnitude of force?

Solution. Magnitude of force, $F = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{49}$ = **7 N**

Example 8.15. **Two vectors are given as $\vec{A} = 3\hat{i} + 9\hat{j} - 6\hat{k}$ and $\vec{B} = 8\hat{i} - 4\hat{j} + 8\hat{k}$. Find $|\vec{A}+\vec{B}|$.**

Solution. $\vec{A}+\vec{B} = (3\hat{i} + 9\hat{j} - 6\hat{k}) + (8\hat{i} - 4\hat{j} + 8\hat{k}) = 11\hat{i} + 5\hat{j} + 2\hat{k}$

$\therefore \quad |\vec{A}+\vec{B}| = \sqrt{11^2 + 5^2 + 2^2} = \sqrt{\mathbf{150}}$

Example 8.16. **The components of a vector $\vec{A}$ are $A_x = 0$; $A_y = 22$. The components of vector $\vec{B}$ are $B_x = 33.2$ and $B_y = -33.2$. Write the vectors (*i*) in unit vector notation and (*ii*) perform the addition.**

Solution. (*i*) In unit vector notation, the vectors will be expressed as :

$$\vec{A} = 0\hat{i} + 22\hat{j} \; ; \; \vec{B} = 33.2\,\hat{i} - 33.2\,\hat{j}$$

(*ii*) $\vec{A}+\vec{B} = (0\,\hat{i} + 22\,\hat{j}) + (33.2\hat{i} - 33.2\,\hat{j}) = (0 + 33.2)\hat{i} + (22 - 33.2)\hat{j} = \mathbf{33.2\,\hat{i} - 11.2\,\hat{j}}$

Example 8.17. **Determine the vector which when added to the resultant of $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives unit vector along *y*-direction.**

Solution. Resultant vector, $\vec{R} = \vec{A} + \vec{B}$

$= (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2\hat{i} + 4\hat{j} - 3\hat{k}) = 5\hat{i} - \hat{j} + 4\hat{k}$

Unit vector along *y*-axis $= \hat{j}$

$\therefore$ Required vector $= \hat{j} - (5\hat{i} - \hat{j} + 4\hat{k}) = \mathbf{-5\hat{i} + 2\hat{j} - 4\hat{k}}$

Example. 8.18. **The *x*- and *y*-components of $\vec{A}$ are 4 and 6 and those of $\vec{A}+\vec{B}$ are 10 and 9. Find the components, magnitude and direction of $\vec{B}$.**

Solution. $\vec{A} = 4\hat{i} + 6\hat{j}$; $\vec{A}+\vec{B} = 10\hat{i} + 9\hat{j}$

$\therefore \quad \vec{B} = (10\hat{i} + 9\hat{j}) - \vec{A} = (10\hat{i} + 9\hat{j}) - (4\hat{i} + 6\hat{j}) = 6\hat{i} + 3\hat{j}$

Therefore, *x*- and *y*-components of $\vec{B}$ are 6 and 3.

$\therefore \quad |\vec{B}| = \sqrt{(6)^2 + (3)^2} = \sqrt{\mathbf{45}}$

Direction of $\vec{B}$, $\tan\theta = \frac{3}{6} = \frac{1}{2} \quad \therefore \quad \theta$ = **26.56° with *x*-axis**

Example 8.19. **Given $\vec{A} = (2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{B} = (3\hat{i} - 5\hat{j} + \hat{k})$. Find the angle between $\vec{A}$ and $\vec{B}$.**

Solution. $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k} \quad \therefore \quad A = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$

$\vec{B} = 3\hat{i} - 5\hat{j} + \hat{k} \quad \therefore \quad B = \sqrt{3^2 + (-5)^2 + 1^2} = \sqrt{35}$

Resultant $\vec{R} = \vec{A} + \vec{B} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + (3\hat{i} - 5\hat{j} + \hat{k})$

$= 5\hat{i} - 2\hat{j} + 5\hat{k}$

$\therefore \quad R = \sqrt{5^2 + 2^2 + 5^2} = \sqrt{54}$

Now $\qquad R^2 = A^2 + B^2 + 2AB\cos\theta$

$\therefore \qquad \cos\theta = \dfrac{R^2 - A^2 - B^2}{2AB} = \dfrac{54 - 29 - 35}{2 \times \sqrt{29} \times \sqrt{35}} = -0.1567 \quad \therefore \quad \theta = 99°$

8.16. MULTIPLICATION OF VECTORS

Vectors have direction as well as magnitude. Therefore, vector multiplication cannot follow exactly the same rules as the algebraic rules of scalar multiplication. Instead, we must *define* the operation of vector multiplication in such a way that we find it useful in physics. It is very useful to define the following two kinds of multiplication operations for vectors :

(*i*) Multiplication of one vector by a second vector so as to produce a scalar. It is called *scalar product* or *dot product* of two vectors.

(*ii*) Multiplication of one vector by a second vector so as to produce another vector. It is called *vector product* or *cross product* of two vectors.

8.17. SCALAR PRODUCT OF TWO VECTORS

Consider two vectors $\vec{A}$ and $\vec{B}$ with angle θ between them as shown in Fig. 8.33. The scalar product of vectors $\vec{A}$ and $\vec{B}$ is defined as :

$$\vec{A}.\vec{B} = AB\cos\theta$$

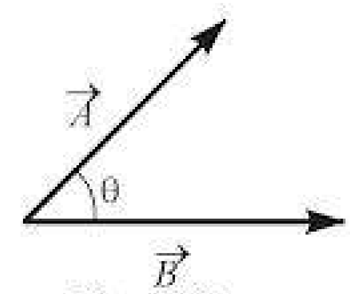


Fig. 8.33

where A and B are the magnitudes of the vectors and θ is the angle between them when their tails touch. Since A, B and cos θ are scalars, the product $\vec{A}.\vec{B}$ (read "A dot B") is *also scalar. The equivalent definition of scalar product is as under :

The **scalar product** *of two vectors* $\vec{A}$ *and* $\vec{B}$ *is defined as the product of magnitude of one vector (say A) and the scalar component of the other vector (B cos θ) along the direction of the first vector* $(\vec{A})$.

Thus in Fig. 8.34 (*i*), $\vec{B}$ has a scalar component $B\cos\theta$ along the direction of $\vec{A}$.

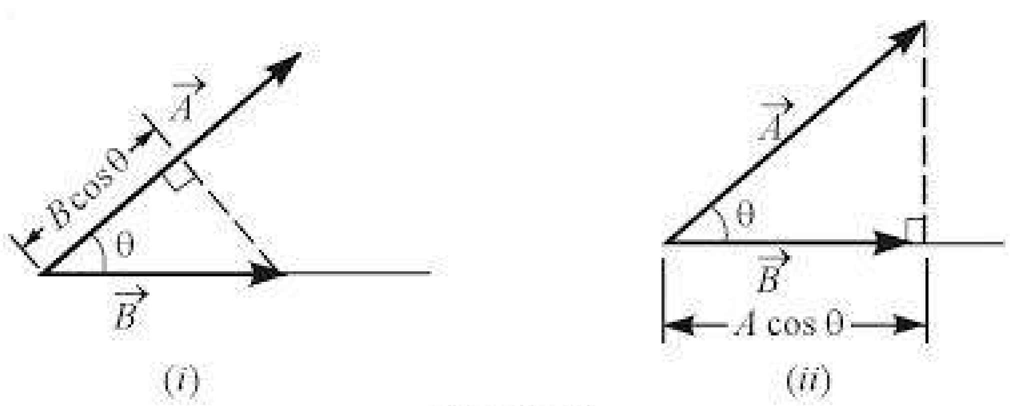


Fig. 8.34

$\therefore \qquad \vec{A}.\vec{B} = (A) \times$ scalar component of $\vec{B}$ along the direction of $\vec{A}$

$\qquad\qquad = (A)\,(B\cos\theta)$

or $\qquad \vec{A}.\vec{B} = AB\cos\theta$

* This definition fits perfectly with our definition of work done by a constant force *i.e.*, we can write the work done by a constant force as the scalar product of force $(\vec{F})$ and displacement $(\vec{S})$.
Work done, $W = \vec{F}.\vec{S} = FS\cos\theta$

Similarly, in Fig. 8.34 (*ii*), $\vec{B}.\vec{A} = (B)\,(A\cos\theta) = AB\cos\theta$

Note that the dot or scalar product of two vectors is a scalar quantity. Each of the vector $\vec{A}$ and $\vec{B}$ has a direction but the scalar product itself does not have a direction.

8.18. PROPERTIES OF SCALAR (OR DOT) PRODUCT

The following properties of scalar product (or dot product) are worth noting :

(*i*) *For given vectors $\vec{A}$ and $\vec{B}$, the value of the scalar product depends upon the angle θ between them* (See Fig. 8.35).

For $\theta = 0°$; $\vec{A}.\vec{B} = AB\cos 0° = AB$

For $\theta = 180°$; $\vec{A}.\vec{B} = AB\cos 180° = -AB$

For $\theta = 90°$; $\vec{A}.\vec{B} = AB\cos 90° = 0$

Thus the dot product of two mutually perpendicular vectors is zero.

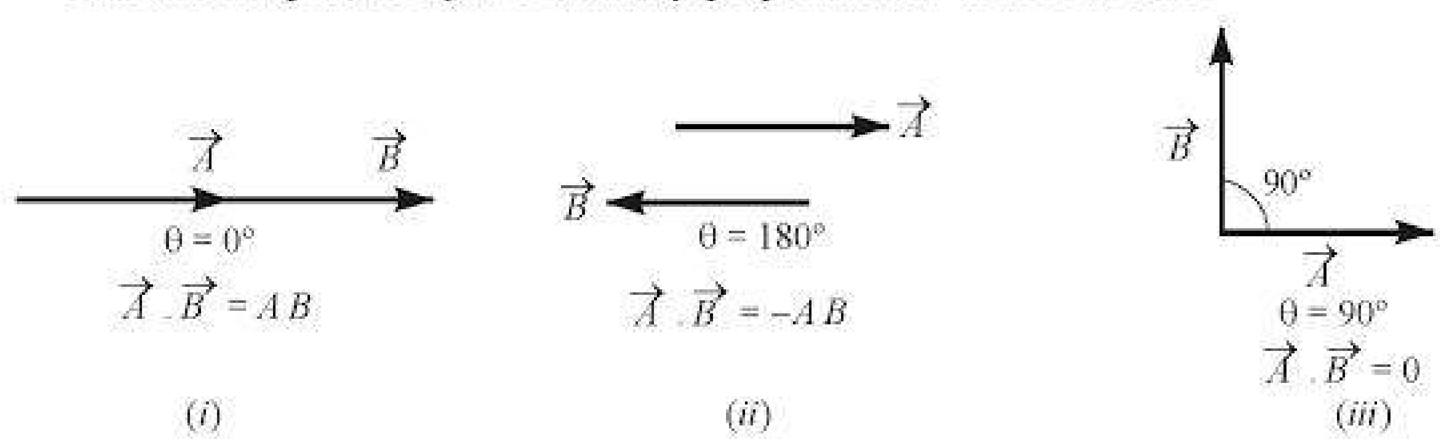


Fig. 8.35

(*ii*) *The dot product of two vectors obeys commutative law.* This directly follows from the definition of dot product.

$$\vec{A}.\vec{B} = A(B\cos\theta) = AB\cos\theta$$

$$\vec{B}.\vec{A} = B(A\cos\theta) = AB\cos\theta$$

$$\therefore \quad \vec{A}.\vec{B} = \vec{B}.\vec{A}$$

This simply means that the order of vectors in the dot product does not matter.

(*iii*) *The dot product obeys the distributive law.* If $\vec{C} = \vec{A} + \vec{B}$, th

$$\vec{D}.\vec{C} = \vec{D}.(\vec{A} + \vec{B})$$

$$= \vec{D}.\vec{A} + \vec{D}.\vec{B}$$

or $$\vec{D}.\vec{C} = \vec{D}.\vec{A} + \vec{D}.\vec{B}$$

We can prove it by referring to Fig. 8.36. The component of vector $\vec{C}$ along the direction of $\vec{D}$ is the sum of the components of vector $\vec{A}$ and $\vec{B}$ along $\vec{D}$.

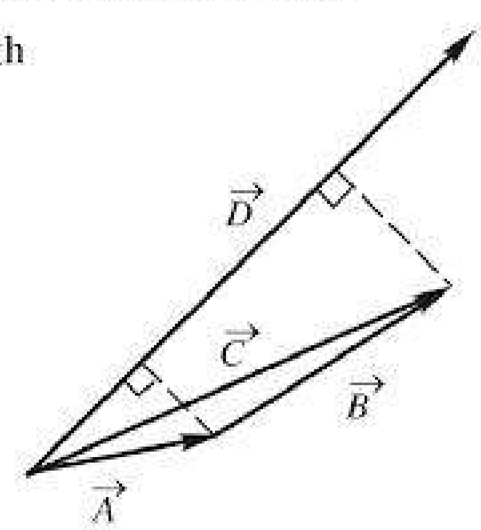


Fig. 8.36

(*iv*) *The dot product of a vector with itself gives square of its magnitude.*

$$\vec{A}.\vec{A} = (A)\,(A)\cos 0° \quad (\theta = 0°) \quad \therefore \quad \vec{A}.\vec{A} = A^2$$

Note. If $\vec{A}$ is perpendicular to $\vec{B}$, then $\vec{A}.\vec{B} = 0$. But if the converse is given *i.e.*, $\vec{A}.\vec{B} = 0$, then there are three possibilities *viz.*, $\vec{A} = 0$, $\vec{B} = 0$ or $\vec{A} \perp \vec{B}$.

Examples of dot product of two vectors. In physics, there are many physical quantities (*e.g.* work, power, magnetic flux *etc.*) that can be described as the scalar or dot product of two vectors.

(*i*) Work done, $W = FS\cos\theta = \vec{F}.\vec{S}$

Thus work done by a constant force is the dot product of force ($\vec{F}$) and displacement ($\vec{S}$).

(*ii*) Instantaneous power, $P = Fv\cos\theta = \vec{F}.\vec{v}$

Thus instantaneous power is the dot product of force ($\vec{F}$) and velocity ($\vec{v}$).

8.19. UNIT VECTORS AND THE DOT PRODUCT

We have seen that if two vectors are aligned (*i.e.*, angle θ between them is zero), their dot product is equal to the product of their magnitudes. However, if the two vectors are perpendicular, their dot product is zero. Using these definitions of dot product, we can derive the following relationships for the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

(*i*) Since $\hat{i}$ is parallel to $\hat{i}$ (*i.e.*, $\theta = 0°$) and each has a unit magnitude,

$\therefore$ $\hat{i}.\hat{i} = (1)(1)\cos 0° = 1$. Similarly, $\hat{j}.\hat{j} = 1$ and $\hat{k}.\hat{k} = 1$.

$\therefore$ $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(*ii*) Since $\hat{i}$ and $\hat{j}$ are perpendicular and each has a unit magnitude,

$\therefore$ $\hat{i}.\hat{j} = (1)(1)\cos 90° = 0$. Similarly, $\hat{i}.\hat{k} = 0$ and $\hat{j}.\hat{k} = 0$.

$\therefore$ $\hat{i}.\hat{j} = \hat{i}.\hat{k} = \hat{j}.\hat{k} = 0$

Consider two three-dimensional vectors $\vec{A}$ and $\vec{B}$. These can be expressed in the rectangular form as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \;;\quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\therefore \quad \vec{A}.\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

With the distributive law, it will yield,

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

This is a very useful relation.

Also $\quad \vec{A}.\vec{B} = AB\cos\theta \quad \therefore \quad \cos\theta = \dfrac{\vec{A}.\vec{B}}{AB} = \dfrac{A_xB_x + A_yB_y + A_zB_z}{AB}$

Thus we can find the angle θ between the vectors $\vec{A}$ and $\vec{B}$. Note that A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively *i.e.*,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \;;\qquad B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Example 8.20. If $\vec{R} = \vec{A} - \vec{B}$, show that $R^2 = A^2 + B^2 - 2AB\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.

Solution. $\vec{R} = \vec{A} - \vec{B}$

Taking dot product of $\vec{R}$ with itself, we have,

$$\vec{R}.\vec{R} = (\vec{A} - \vec{B}).(\vec{A} - \vec{B})$$

$$= \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B}$$

or $$R^2 = A^2 - AB\cos\theta - BA\cos\theta + B^2$$

$\therefore$ $$R^2 = A^2 + B^2 - 2AB\cos\theta$$

Example 8.21. Find the angle between the vectors $\vec{A} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{B} = 5\hat{i} - 2\hat{j} - 3\hat{k}$.

Solution. $$\vec{A}.\vec{B} = AB\cos\theta \quad \therefore \quad \cos\theta = \frac{\vec{A}.\vec{B}}{AB}$$

Now $$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z = 3 \times 5 + 2\,(-2) + (1)\,(-3) = 8$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(5)^2 + (-2)^2 + (-3)^2} = \sqrt{38}$$

$\therefore$ $$\cos\theta = \frac{8}{\sqrt{14 \times 38}} = 0.35 \text{ or } \theta = \cos^{-1} 0.35 = 69.5°$$

Example 8.22. The sum and difference of two vectors are perpendicular to each other. Prove that the vectors are equal in magnitude.

Solution. Suppose the two vectors are $\vec{A}$ and $\vec{B}$. The sum and difference are $(\vec{A}+\vec{B})$ and $(\vec{A}-\vec{B})$. Since these two are perpendicular to each other, their dot product is zero *i.e.*,

$$(\vec{A}+\vec{B}).(\vec{A}-\vec{B}) = 0 \text{ or } \vec{A}.\vec{A} - \vec{A}.\vec{B} + \vec{B}.\vec{A} - \vec{B}.\vec{B} = 0$$

or $$A^2 - AB\cos\theta + BA\cos\theta - B^2 = 0 \quad \text{or} \quad A^2 - B^2 = 0 \quad \therefore \; A = \pm B$$

Example 8.23. The sum and difference of two vectors are equal in magnitude *i.e.* $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$. Prove that vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Solution. $$|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$$

or $$|\vec{A}+\vec{B}|^2 = |\vec{A}-\vec{B}|^2 \qquad \ldots (i)$$

We know that dot product of a vector with itself is equal to the square of the magnitude of the vector. Therefore, expression given in eq. (*i*) can be written as :

$$(\vec{A}+\vec{B}).(\vec{A}+\vec{B}) = (\vec{A}-\vec{B}).(\vec{A}-\vec{B})$$

or $$\vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B} = \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B}$$

or $$A^2 + AB\cos\theta + AB\cos\theta + B^2 = A^2 - AB\cos\theta - AB\cos\theta + B^2$$

or $$2AB\cos\theta = -2AB\cos\theta$$

or $$4AB\cos\theta = 0 \text{ or } \cos\theta = 0 \;\therefore\; \theta = 90°$$

Example 8.24. If the magnitudes of two vectors are 3 and 4 and the magnitude of their scalar product is 6, find the angle between the vectors.

Solution. Let θ be the angle between the two vectors $\vec{A}$ and $\vec{B}$. Their scalar product is

$$\vec{A}.\vec{B} = AB\cos\theta$$

Here $\quad \vec{A}.\vec{B} = 6\ ;\ A = 3\ ;\ B = 4$

$\therefore \quad 6 = 3 \times 4 \times \cos\theta \quad$ or $\quad \cos\theta = \dfrac{6}{3 \times 4} = 0.5 \quad \therefore \quad \theta = 60°$

Example 8.25. Prove that vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j}$ are perpendicular to each other.

Solution. $\quad \vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}\ ;\quad \vec{B} = 2\hat{i} - \hat{j}$

The two vectors are perpendicular if $\vec{A}.\vec{B} = 0$.

Now $\quad \vec{A}.\vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k}).(2\hat{i} - \hat{j})$

$$= 1 \times 2 + 2 \times (-1) = 2 - 2 = 0$$

Since the dot product of $\vec{A}$ and $\vec{B}$ is zero, the two vectors are mutually perpendicular.

Example 8.26. The resultant vector of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. On reversing the direction of $\vec{Q}$, the resultant vector becomes $\vec{S}$. Show that :

$$\boldsymbol{R^2 + S^2 = 2(P^2 + Q^2)}$$

Solution. Let θ be the angle between $\vec{P}$ and $\vec{Q}$.

Given : $\quad \vec{R} = \vec{P} + \vec{Q}$ and $\vec{S} = \vec{P} - \vec{Q}$

Now $\quad \vec{R}.\vec{R} = (\vec{P} + \vec{Q}).(\vec{P} + \vec{Q})$

or $\quad R^2 = \vec{P}.\vec{P} + \vec{P}.\vec{Q} + \vec{Q}.\vec{P} + \vec{Q}.\vec{Q}$

or $\quad R^2 = P^2 + PQ\cos\theta + PQ\cos\theta + Q^2$

or $\quad R^2 = P^2 + Q^2 + 2PQ\cos\theta \qquad \ldots (i)$

Also $\quad \vec{S}.\vec{S} = (\vec{P} - \vec{Q}).(\vec{P} - \vec{Q})$

or $\quad S^2 = \vec{P}.\vec{P} - \vec{P}.\vec{Q} - \vec{Q}.\vec{P} + \vec{Q}.\vec{Q}$

or $\quad S^2 = P^2 - PQ\cos\theta - PQ\cos\theta + Q^2$

or $\quad S^2 = P^2 + Q^2 - 2PQ\cos\theta \qquad \ldots (ii)$

Adding eqs. (*i*) and (*ii*), we have,

$$R^2 + S^2 = 2(P^2 + Q^2)$$

Example 8.27. A particle moves from position $(3\hat{i} + 2\hat{j} - 6\hat{k})$ to a position $(14\hat{i} + 13\hat{j} + 9\hat{k})$ in metre units and a constant force $(4\hat{i} + 2\hat{j} + 3\hat{k})$ newton acts on it. Calculate the work done by the force.

Solution. $\vec{r_1} = 3\hat{i} + 2\hat{j} - 6\hat{k}$; $\vec{r_2} = 14\hat{i} + 13\hat{j} + 9\hat{k}$

Displacement of the particle, $\vec{S} = \vec{r_2} - \vec{r_1} = (14\hat{i} + 13\hat{j} + 9\hat{k}) - (3\hat{i} + 2\hat{j} - 6\hat{k})$

$= 11\hat{i} + 11\hat{j} + 15\hat{k}$

The work done by a constant force is equal to the dot product of force and displacement.

$\therefore$ Work done, $W = \vec{F}.\vec{S} = (4\hat{i} + 2\hat{j} + 3\hat{k}).(11\hat{i} + 11\hat{j} + 15\hat{k})$

$= 4\times 11 + 2\times 11 + 3\times 15 = 44 + 22 + 45 =$ **111 J**

Example 8.28. If unit vectors $\hat{A}$ and $\hat{B}$ are inclined at an angle θ, then prove that :

$$|\hat{A} - \hat{B}| = 2\sin\frac{\theta}{2}$$

Solution. We know that $|\hat{A} - \hat{B}|^2 = (\hat{A} - \hat{B}).(\hat{A} - \hat{B})$

or $|\hat{A} - \hat{B}|^2 = \hat{A}\cdot\hat{A} + \hat{B}\cdot\hat{B} - 2\hat{A}.\hat{B}$

$= (1)^2 + (1)^2 - 2(1)(1)\cos\theta$

$= 2 - 2\cos\theta = 2(1 - \cos\theta)$

$= 2\left[1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right] = 4\sin^2\frac{\theta}{2}$

$\therefore$ $|\hat{A} - \hat{B}| = 2\sin\frac{\theta}{2}$

Example 8.29. Find the magnitude and direction of vector $\hat{i} + \hat{j}$.

Solution. Magnitude of vector $\hat{i} + \hat{j}$ is

$|\hat{i} + \hat{j}| = [(1)^2 + (1)^2]^{1/2} = \sqrt{2}$

Suppose the vector $\hat{i} + \hat{j}$ makes an angle θ with x-axis (*i.e.* with unit vector $\hat{i}$). Then,

$(\hat{i} + \hat{j}).\hat{i} = |\hat{i} + \hat{j}|\,|\hat{i}|\cos\theta$ $\quad(\because \vec{A}.\vec{B} = AB\cos\theta)$

or $\hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{i} = (\sqrt{2})(1)\cos\theta$

or $1 + 0 = \sqrt{2}\cos\theta$ $\therefore$ $\cos\theta = \frac{1}{\sqrt{2}}$ or $\theta =$ **45°**

Example 8.30. Find the unit vector parallel to the resultant of vectors $\vec{A} = 2\hat{i} - 6\hat{j} - 3\hat{k}$ and $\vec{B} = 4\hat{i} + 3\hat{j} - \hat{k}$.

Solution. Let $\vec{R}$ be the resultant of $\vec{A}$ and $\vec{B}$.

$\vec{R} = \vec{A} + \vec{B} = (2\hat{i} - 6\hat{j} - 3\hat{k}) + (4\hat{i} + 3\hat{j} - \hat{k}) = 6\hat{i} - 3\hat{j} - 4\hat{k}$

$\therefore$ $R = |\vec{R}| = \sqrt{6^2 + (-3)^2 + (-4)^2} = \sqrt{61}$

The unit vector parallel to $\vec{R}$ is given by ;

$$\hat{R} = \frac{\vec{R}}{R} = \frac{6\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{61}} = \frac{6}{\sqrt{61}}\hat{i} - \frac{3}{\sqrt{61}}\hat{j} - \frac{4}{\sqrt{61}}\hat{k}$$

PROBLEMS FOR PRACTICE

1. Find the angle between the X-axis and the vector $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$. $\left[\cos^{-1}\frac{2}{\sqrt{29}}\right]$
2. Find the angle between the vectors $\vec{A} = 2\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 2\hat{j} - 4\hat{k}$. **[90°]**
3. The magnitudes of two vectors $\vec{A}$ and $\vec{B}$ are 5 and 4 units respectively and the angle between them is 30°. Find the value of $\vec{A}.\vec{B}$. **[17.32 units]**
4. If $\vec{A} + \vec{B} = \vec{C}$ and $A^2 = B^2 + C^2$, then prove that $\vec{A}$ and $\vec{B}$ are perpendicular to each other.
5. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non zero vectors and $\vec{A}.\vec{B} = 0$ and $\vec{B}.\vec{B} = 0$, find the value of $\vec{A}.\vec{C}$. **[AC]**
6. Determine the value of a so that vectors $\vec{A} = 5\hat{i} + 7\hat{j} - 3\hat{k}$ and $\vec{B} = 2\hat{i} + 2\hat{j} - a\hat{k}$ are perpendicular. **[– 8]**
7. Prove that vectors $\vec{A} = -2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other.
8. A point of application of force $\vec{F} = 5\hat{i} - 3\hat{j} + 2\hat{k}$ is moved from $\vec{r_1} = 2\hat{i} + 7\hat{j} + 2\hat{k}$ to $\vec{r_2} = -5\hat{i} + 2\hat{j} + 3\hat{k}$. Find the work done. **[– 22 units]**
9. Find the angle θ between vectors $\vec{A}$ and $\vec{B}$ where $\vec{A} = -2\hat{i} + 2\hat{j}$ and $\vec{B} = 2\sqrt{3}\,\hat{i} + 2\hat{j}$. **[105°]**
10. Show that $\vec{A}$ is perpendicular to $\vec{B}$ if $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$.

8.20. VECTOR PRODUCT

Consider two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. 8.37 (*i*) where the angle θ is the smaller of the angles between the two vectors. The vector or cross product of vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$ given by :

$$\vec{C} = \vec{A} \times \vec{B}$$

It is called vector product of vectors $\vec{A}$ and $\vec{B}$ because $\vec{C}$ is itself a vector. The magnitude and direction of $\vec{C}(= \vec{A} \times \vec{B})$ are defined as under :

(*i*) The magnitude of $\vec{C}$ is given by : $C = AB\sin\theta$

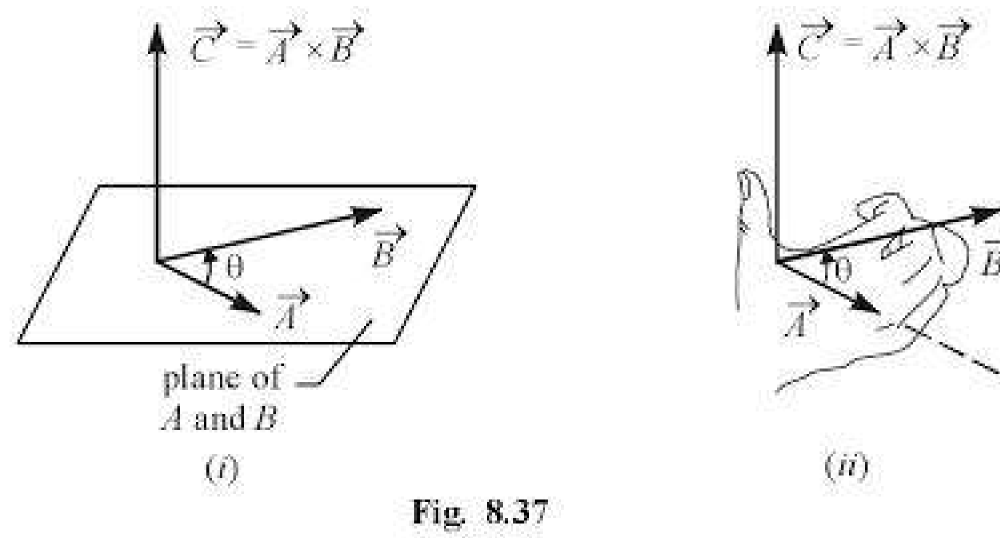


Fig. 8.37

Here A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ while θ is the smaller of the angles between vectors $\vec{A}$ and $\vec{B}$ [See Fig. 8.37 (*i*)].

(*ii*) The vector $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and its direction is given by **right-hand rule** [See Fig. 8.37 (*ii*)]. **Curl the fingers of the right hand in the sense that would rotate the first vector $\vec{A}$ into the second vector $\vec{B}$ through the smaller angle between them. The extended thumb gives the direction of $\vec{C}\ (= \vec{A} \times \vec{B})$.** Applying this rule to Fig. 8.37(*ii*), the direction of $\vec{C}$ is vertically upward.

Note. The direction of $\vec{C}\ (= \vec{A} \times \vec{B})$ can also be found by **Right hand screw rule** stated below :

If a right handed screw placed with its axis perpendicular to the plane containing the two vectors is rotated from $\vec{A}$ to $\vec{B}$ through smaller angle, then the direction of advance of the tip of the screw gives the direction of $\vec{C}\ (= \vec{A} \times \vec{B})$.

Applying this rule to Fig. 8.37(*ii*), the direction of $\vec{C}$ is vertically upward.

8.21. PROPERTIES OF VECTOR PRODUCT

The following are the important properties of vector product or cross product :

(*i*) *For given vectors $\vec{A}$ and $\vec{B}$, the value of the cross product for angle 90° between them is equal to the product of magnitudes of the two vectors i.e.,*

For $\theta = 90°$, $\vec{A} \times \vec{B} = AB \sin 90° = AB$

(*ii*) *The cross product of two vectors does not obey commutative law i.e.,* $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$. Thus referring to Fig. 8.38, the direction of vector $\vec{A} \times \vec{B}$ is opposite to that of vector $\vec{B} \times \vec{A}$. Since the magnitude in each case is $AB \sin\theta$, it follows, therefore, that

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$

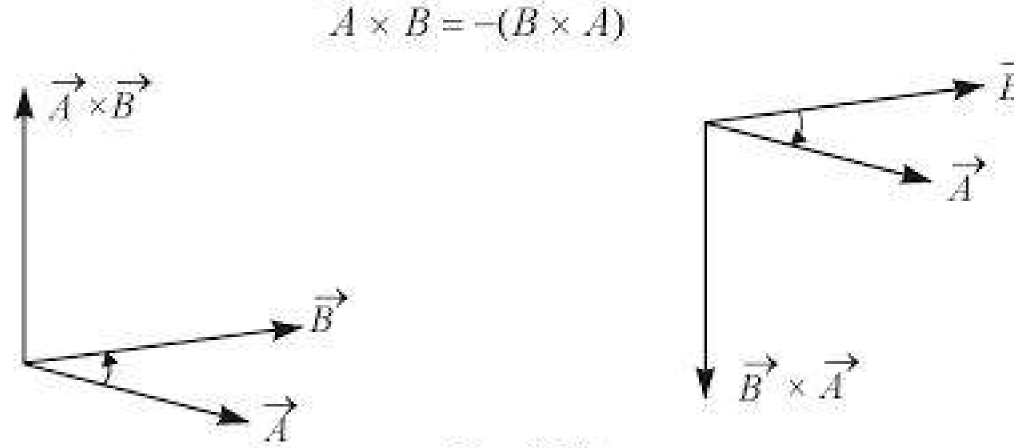


Fig. 8.38

Thus reversing the order of the vectors reverses the direction of the cross product. Therefore, commutative law does not hold for the cross product although it does for the dot product of two vectors.

(*iii*) *The cross product of a vector with itself is zero i.e.,*

$$\vec{A} \times \vec{A} = 0 \qquad (\because\ \theta = 0°)$$

(*iv*) Suppose two vectors $\vec{A}$ and $\vec{B}$ are parallel or antiparallel. The angle θ between them is either 0° or 180°. Then $\vec{A} \times \vec{B} = 0$. *The cross product of parallel (or antiparallel vectors) is zero.*

(*v*) *The cross product obeys distributive law i.e.,*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(*vi*) *The magnitude of the cross product of two vectors is equal to the area of parallelogram formed by them.*

Suppose two vectors $\vec{A}$ and $\vec{B}$ are represented in magnitude and direction by the two adjacent sides $\overrightarrow{OM}$ and $\overrightarrow{OL}$ of the parallelogram *OLKM* (See Fig. 8.39). The area of the parallelogram is given by :

$$= OL \times MN = B(A \sin\theta) = AB \sin\theta$$

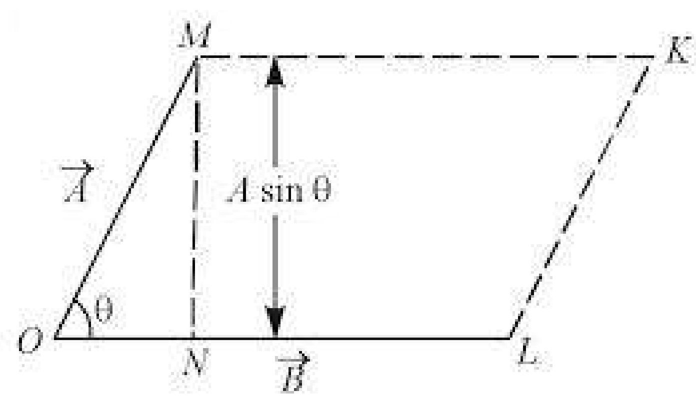


Fig. 8.39

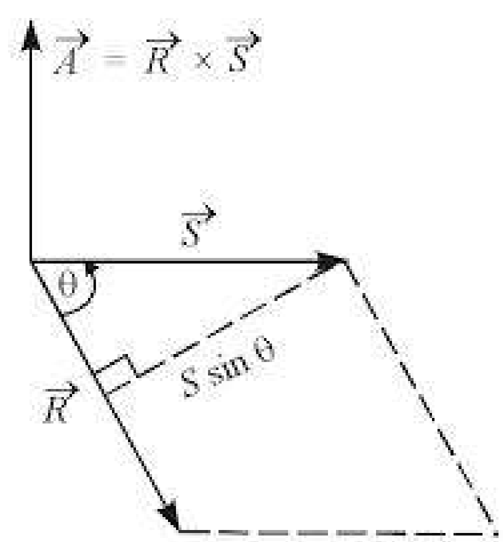


Fig. 8.40

But $AB \sin\theta$ is the magnitude of the cross product $\vec{A} \times \vec{B}$. Therefore, the magnitude of the cross product of two vectors is equal to the area of the parallelogram formed by them.

Note. The area of parallelogram formed by the two vectors is

Area *OLKM* = 2 × Area of triangle formed by two vectors

Hence the magnitude of the cross product of two vectors is equal to twice the area of triangle formed by them.

(*vii*) There is an interesting and useful geometrical interpretation of the cross product of two displacements. The displacements $\vec{R}$ and $\vec{S}$ in Fig. 8.40 form a parallelogram whose area $A = RS \sin\theta$.

This area is the magnitude of the cross product of $\vec{R}$ and $\vec{S}$ *i.e.*, $A = |\vec{R} \times \vec{S}| = RS \sin\theta$.

We define an oriented area element as a vector $\vec{A} = \vec{R} \times \vec{S}$ *whose magnitude is the area of the parallelogram formed by* $\vec{R}$ *and* $\vec{S}$ *and whose direction is perpendicular to plane of* $\vec{R}$ *and* $\vec{S}$. In this way, we specify the orientation of an element of surface area.

Examples of cross product of two vectors. In physics, there are many physical quantities (*i.e.* torque, linear velocity, centripetal acceleration *etc.*) that can be described as the vector or cross product of two vectors.

(*i*) Linear velocity, $\vec{v} = \vec{\omega} \times \vec{r}$

Thus linear velocity of a particle in rotational motion is equal to the cross product of its angular velocity $(\vec{\omega})$ and displacement vector $(\vec{r})$.

(*ii*) Centripetal acceleration, $\vec{a}_c = \vec{\omega} \times \vec{v}$

Thus the centripetal acceleration of a particle in rotational motion is equal to the cross product of its angular velocity $(\vec{\omega})$ and its linear velocity $(\vec{v})$.

8.22. UNIT VECTORS AND THE CROSS PRODUCT

The cross product of two vectors can be evaluated in terms of the rectangular components. To do so, we must consider the cross product of the unit vectors.

(*i*) Let us evaluate $\hat{i}\times\hat{i}$. The result is zero. It is because the two vectors are parallel ($\sin\theta = 0$) and each has a unit magnitude.

$\therefore$ $\hat{i}\times\hat{i} = (1)\,(1)\,\sin 0° = 0$

Similarly, $\hat{j}\times\hat{j}=0$ and $\hat{k}\times\hat{k}=0$

$\therefore$ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

(*ii*) To evaluate $\hat{i}\times\hat{j}$, refer to Fig. 8.41 which shows the unit vectors on XYZ – *coordinate system. The magnitude of each unit vector is 1 and so the magnitude of $\hat{i}\times\hat{j}$ is 1 *i.e.*, $|\hat{i}\times\hat{j}| = (1)\,(1)\,\sin 90° = 1$. The direction of $\hat{i}\times\hat{j}$ is given by the right-hand rule and from Fig. 8.41, it is along the positive Z-axis. But this is just the unit vector $\hat{k}$. Therefore, we have $\hat{i}\times\hat{j}=\hat{k}$. Simply reversing the order of the unit vectors gives $\hat{j}\times\hat{i}=-\hat{k}$.

$\therefore$ $\hat{i}\times\hat{j}=-(\hat{j}\times\hat{i})=\hat{k}$

Similarly, $\hat{j}\times\hat{k}=-(\hat{k}\times\hat{j})=\hat{i}$

and $\hat{k}\times\hat{i}=-(\hat{i}\times\hat{k})=\hat{j}$

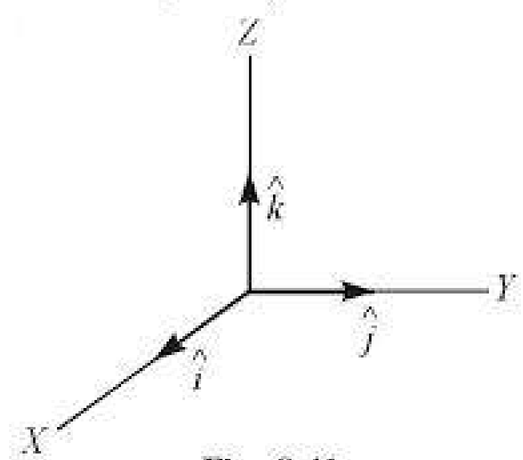


Fig. 8.41

Consider three dimensional vectors $\vec{A}$ and $\vec{B}$. These vectors can be expressed in terms of rectangular vectors as:

$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\,;\quad \vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$$

We form cross product $\vec{C}=\vec{A}\times\vec{B}$, expressing $\vec{A}$ and $\vec{B}$ in terms of their components.

$$\therefore\quad \vec{A}\times\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$

The distributive property allows us to carry out the multiplication and we get,

$$\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k} \quad ...(i)$$

In determinant form, this can be written as :

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Another way to express the result is

$$\vec{A}\times\vec{B}=\vec{C}=C_x\hat{i}+C_y\hat{j}+C_z\hat{k} \quad ...(ii)$$

* The coordinate system shown in Fig. 8.41 is called *right-handed coordinate system*. The unit vectors $\hat{i},\hat{j},\hat{k}$ associated with the X, Y and Z axes in that order are connected by the right-hand rule $\hat{i}\times\hat{j}=\hat{k}$.

By comparison with eq. (i), the components of $\vec{C}$ are :

$$C_x = A_yB_z - A_zB_y\ ;\ C_y = A_zB_x - A_xB_z\ ;\ C_z = A_xB_y - A_yB_x$$

Example 8.31. Find the cross product of vectors $\vec{A} = 2\hat{i}$ and $\vec{B} = -2\hat{i} + 4\hat{j}$.

Solution.

$$\vec{A} \times \vec{B} = (2\hat{i}) \times (-2\hat{i} + 4\hat{j})$$

$$= -4(\hat{i} \times \hat{i}) + 8(\hat{i} \times \hat{j})$$

$$= 0 + 8\hat{k} = \mathbf{8\hat{k}}$$

Note that in this example we see again that the plane of $\vec{A}$ and $\vec{B}$ (the $\hat{i} - \hat{j}$ plane) is perpendicular to the direction of $\vec{A} \times \vec{B}$ (the $\hat{k}$ direction).

Example 8.32. Determine the area of the parallelogram whose adjacent sides are $2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} - \hat{j}$.

Solution. Let $\vec{A}$ and $\vec{B}$ be the vectors representing the adjacent sides of the parallelogram.

Here $\vec{A} = 2\hat{i} + \hat{j} + 3\hat{k}\ ;\quad \vec{B} = \hat{i} - \hat{j}$

The area of parallelogram is equal to the magnitude of the cross product of $\vec{A}$ and $\vec{B}$.

Now

$$\vec{A} \times \vec{B} = (2\hat{i} + \hat{j} + 3\hat{k}) \times (\hat{i} - \hat{j})$$

$$= 2(\hat{i} \times \hat{i}) - 2(\hat{i} \times \hat{j}) + (\hat{j} \times \hat{i}) - (\hat{j} \times \hat{j}) + 3(\hat{k} \times \hat{i}) - 3(\hat{k} \times \hat{j})$$

$$= 0 - 2\hat{k} - \hat{k} - 0 + 3\hat{j} + 3\hat{i}$$

$\therefore$

$$\vec{A} \times \vec{B} = 3\hat{i} + 3\hat{j} - 3\hat{k}$$

$\therefore$ Magnitude of the area of the parallelogram

$$= \sqrt{(3)^2 + (3)^2 + (-3)^2} = \sqrt{27} = \mathbf{3\sqrt{3}}\ \textbf{sq units}$$

Example 8.33. Show that vectors $\vec{A} = \hat{i} - 5\hat{j}$ and $\vec{B} = 2\hat{i} - 10\hat{j}$ are parallel to each other.

Solution. If the two vectors are parallel, then their cross product must be zero *i.e.*, $\vec{A} \times \vec{B} = 0$.

Now

$$\vec{A} \times \vec{B} = (\hat{i} - 5\hat{j}) \times (2\hat{i} - 10\hat{j})$$

$$= 2(\hat{i} \times \hat{i}) - 10(\hat{i} \times \hat{j}) - 10(\hat{j} \times \hat{i}) + 50(\hat{j} \times \hat{j})$$

$$= 2(0) - 10(\hat{k}) - 10(-\hat{k}) + 50(0) = 0$$

Hence vectors $\vec{A}$ and $\vec{B}$ are parallel to each other.

Example 8.34. Find a unit vector perpendicular to both the vectors $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Solution. By definition, the cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$.

Now

$$\vec{A} \times \vec{B} = (3\hat{i} + \hat{j} + 2\hat{k}) \times (2\hat{i} - 2\hat{j} + 4\hat{k}) = 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$\therefore$

$$|\vec{A} \times \vec{B}| = \sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$$

∴ Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \frac{\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}}{\sqrt{\mathbf{3}}}$$